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THESIS

BUCKLING NEAR A HOLE IN AN INFINITE PLATE

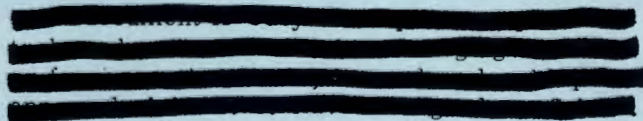
UNDER TENSION

by

Robert Graham Costello

June 1968

THESIS
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BUCKLING NEAR A HOLE IN AN INFINITE PLATE

UNDER TENSION

by

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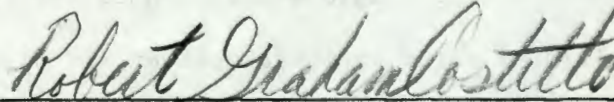
Submitted in partial fulfillment of the
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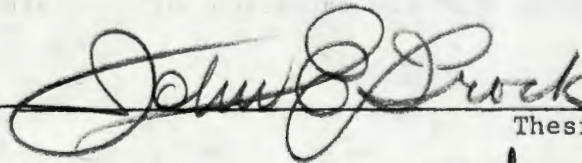
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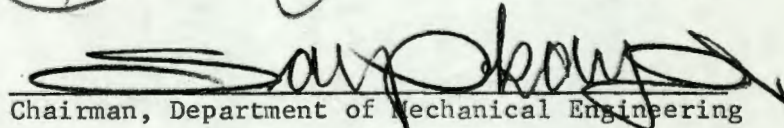
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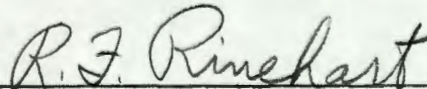
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ABSTRACT

If an infinite flat elastic plate containing a circular hole is subjected to a loading which amounts to simple uniaxial tension at distances remote from the hole, the classical solution by Kirsch provides an evaluation of the stress components throughout the plate, provided the loading is sufficiently small. However if the tensile loading is progressively increased, there comes a point when Kirsch's solution becomes invalid, either through inelastic action at the most highly stressed regions in the plate, or through buckling of the plate from its original plane. The question of buckling under these circumstances has previously been discussed only by Danis, who dealt experimentally with finite plates, and by Pellett who performed a theoretical study of an infinite plate. The present thesis pinpoints and corrects some errors in Pellett's analysis, and leads to the result that buckling impends when the tensile stress reaches the value

$$S_{cr} = 1.720 E (t/a)^2$$

where E denotes Young's modulus and t/a denotes the ratio of plate thickness to hole radius. This evaluation is for Poisson's ratio $\nu = 0.3$, a commonly used value, but evaluations are also made for other values of ν , indicating that the variation with respect to ν is quite small.

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NOMENCLATURE

a	Hole radius
A_{nm}	A set of unknown coefficients in a series describing plate deflection
$\underline{A}$	An eigenvector
b	The half-width of an infinite strip
$\underline{\underline{B}}$	A coefficient matrix (Appendix B)
$\underline{\underline{C}}$	A coefficient matrix (Appendix C)
d	Hole diameter
D	Flexural rigidity of the plate = $Eh^3/12 (1 - \nu^2)$
E	Young's Modulus
h	Ratio of plate thickness to hole radius, (t/a)
$\underline{\underline{I}}$	Identity matrix
I	A dummy integer index for the index n
I'	A single subscript representing the double subscript IJ
I_1	Integral in numerator of equation 2
I_2	Integral in denominator of equation 2
J	A dummy integer index for the index m
J'	A single subscript representing the double subscript KL
$\bar{J}$	Number of unknown coefficients = $M \cdot N$
K	A dummy integer index for the index n
K	Buckling coefficient
L	A dummy integer index for the index m
m	An integer index
M	Number of terms in summation on m
M_r	Bending moment per unit length
M_{rt}	Twisting moment per unit length

NOMENCLATURE (Continued)

n	An integer index
N	Upper limit of summation on n
$N_x, N_y,$ N_{xy}	Membrane forces per unit length = $\sigma_x h$, etc.
P_{IJKL}	A term in the quadruple series that expresses the bending energy integral
$\underline{P}$	The matrix of elements P_{IJKL}
Q_{IJKL}	A term in the quadruple series that expresses the membrane energy integral
$\underline{Q}$	The matrix of elements Q_{IJKL}
r	Distance from the hole center, also a coordinate of the cylindrical system
$\underline{R}$	A matrix of the eigenvectors of $\underline{P}$
$s_1, s_2,$ s_3	Intermediate functions of N, M , and ν (Appendix A)
S	Externally applied stress
S_{cr}	The value of S for incipient buckling
$t_1, t_2,$ t_3	Intermediate functions of N, M , and ν (Appendix A)
t	Plate thickness
u	Component of displacement in the x direction
v	Component of displacement in the y direction
w	Component of displacement in the z direction (lateral deflection)
$w_r, w_{rr},$ etc.	Derivatives of plate deflection with respect to the variables indicated by subscripts
x	A coordinate in the cartesian system
X_{nm}	A coefficient which is used to satisfy the boundary conditions for each term in n or m

NOMENCLATURE (Continued)

<u>X</u>	An eigenvector (Appendix B)
y	A coordinate of the cartesian system; also a scalar parameter in an eigenvalue problem (Appendix B)
Y_{nm}	A coefficient which is used to satisfy the boundary conditions for each term in n or m
<u>Y</u>	Vector of coefficients in Appendix C
δ	The variation operator, (also see equation 20)
ϵ	The sum of the squares of errors (Appendix C)
Θ	An angular coordinate in the cylindrical system
λ	A scalar parameter in the eigenvalue problem
ν	Poisson's ratio
$\sigma_x, \sigma_y, \sigma_r, \sigma_\theta$	Normal stress components
τ_{xy}, τ_{re}	Shear stress components
Φ	A stress function (Appendix D)

OTHER NOTATION

A double underline denotes a square matrix. (A)

A single underline denotes a column vector. (A)

T (as a superscript) denotes transposition of a matrix. (A^T)

-1 (as a superscript) denotes inversion of a nonsingular matrix. (A⁻¹)

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1. Introduction.

The well known solution by Kirsch (6) for the membrane stresses in an infinite plate containing a circular hole and under a state of uniaxial tensile stress at an infinite distance shows that there are regions in the plate where one or both of the principal membrane stresses are compressive. Thus it seems reasonable to expect that for sufficiently thin plates under sufficiently great tensile stress, the plate may buckle out of its original plane. Danis (2) and Pellett (7) seem to be the only ones who have studied this problem. Danis performed tests on plates of finite dimensions and inferred, incorrectly we now believe, a certain type of dependence of buckling stress upon ratio of plate width to hole diameter. Pellett considered the problem of the infinite plate, using an energy method to calculate the critical buckling stress. However, his analysis contains some errors which invalidate his results. These errors came to attention during attempts to deal with "Howland's problem", described in Appendix D hereof, and attention was immediately focussed upon correcting Pellett's analysis and computing corrected results. Grateful acknowledgment is made to Pellett for his cooperation in verifying the nature of his errors and in offering suggestions for correcting the analysis.

During the recomputations which were necessary, using the corrected theory, two additional improvements were made on what Pellett had done earlier. For one thing, the nature of the analysis requires that the buckling stress be inferred as a lower bound of a number of determinations each of which is itself slightly in error. In the present analysis a least-squares inference is made so as to maximize the objectivity of the inference. Also, although it might be expected that variation in Poisson's ratio would have but slight influence upon the buckling stress, a quantitative comparison is of interest, and this is accomplished herein.

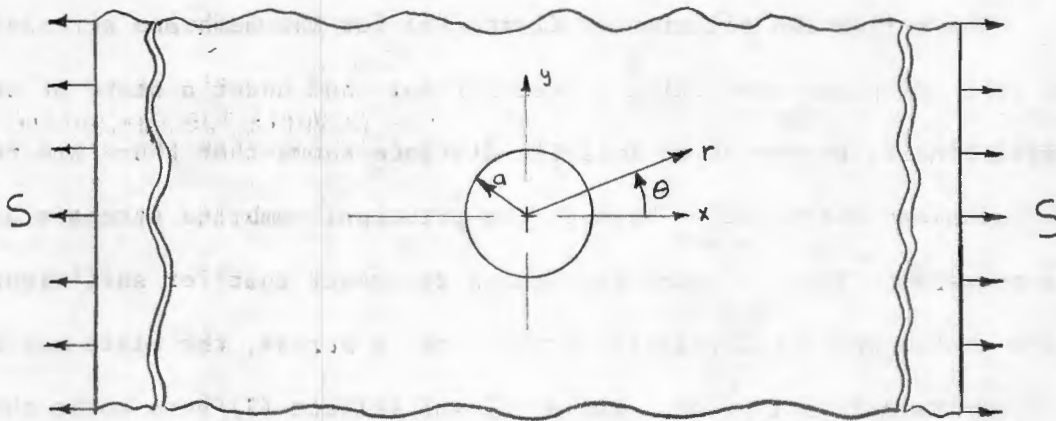


Figure 1

The specific problem under investigation is illustrated in Figure 1. The center of the circular hole is taken as the origin of a set of cylindrical coordinates (r, θ, z) and a set of cartesian coordinates (x, y, z) . The z -axis, not shown, forms a right-handed orthogonal set with the x and y axes. The radius of the hole is a (See Figure 1) and the plate thickness is t . For convenience we temporarily take the unit of length to be one hole radius, so that the corresponding hole radius is unity and the plate thickness is $h = t/a$. However, the influence of variations in hole radius is properly re-introduced in reporting final results.

At a very large distance from the hole it is assumed that the state of stress is a simple uniaxial tensile stress S in the x direction. If the value of S is sufficiently small, Kirsch's solution correctly describes the stress distribution in the entire plate. However, if S is increased, Kirsch's solution will eventually become invalid, either through inelastic action of engineering materials stressed above their proportional limits, or through buckling of the plate out of its original plane.

Kirsch's solution, along with a knowledge of the properties and behavior of the material, is sufficient to permit determining the value of S for which inelastic action begins. The problem of interest herein is determining the value of S for which buckling is imminent, under the presumption that the material is still perfectly elastic.

This problem will be called the problem of incipient buckling. The critical value of S is that above which Kirsch's solution is no longer valid because of buckling. However, no attempt is made herein to discuss the post-buckling problem, that is, the problem of determining stress and deflection distributions for values of S greater than that required for incipient buckling. It is clear, however, that information about this evidently difficult problem would be of value.

Similarly, since all plates in existence are finite in all dimensions, it would be of interest to consider finite plates. The infinite plate is an evident good case with which to start because, besides offering fewer mathematical difficulties than the infinitely long strip of finite width (hereafter called the infinite strip) or the finite plate, there is available a tractable solution for membrane stresses. Before the errors in Pellett's analysis came to attention, the writer was attempting to deal with the problem of the infinite strip. Although no useful results were obtained for this much more difficult problem, nevertheless some details of the attempts are reported in Appendix D for whatever use they may be to others who will work on this problem.

It is proper to delineate the relation of this thesis to that of Pellett. The viewpoints are essentially the same. The fundamental equations from which the analysis proceeds are essentially the same except that Pellett made an error in his derivation which amounts to a factor of two. In the present thesis no derivation is made, the appropriate equations being

taken directly from a standard reference (8). Most of the remainder of the analysis is essentially the same in the two theses, except for differences in notation, a slight simplification of equations occurring in Appendix A, and the early introduction, in the present thesis, of matrix notation. Many of the details in the developments are so nearly identical that the present thesis does not include certain tabular (and similar) listings and presentations given by Pellett, the reason being that if a reader requires these details and is unable to reconstruct them easily for himself, he may find them in Pellett's thesis. Furthermore, as mentioned earlier, the present thesis contains an evaluation of the influence of variations in Poisson's ratio and a least-squares extrapolation for the lowest critical stress (see Appendix C), neither of which appears in Pellett's work.

2. Analytical Procedure.

Timoshenko and Gere (8) discuss two methods of solving problems such as the one posed here. One method is to seek solutions of the equilibrium equation *

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left[N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (1)$$

where N_x, N_y , and N_{xy} are membrane forces, w is the lateral deflection, and D is the plate flexural rigidity, $D = Eh^3/12(1-\nu^2)$. This equation is to be solved for w , subject to certain boundary conditions on w .

Timoshenko and Gere also discuss what they call the energy method. Because of the evident mathematical difficulties which would be encountered in attempting to use the first method for dealing with the present buckling problem, it was decided to use the second method, that is, the energy method which is contained in the following equation: (equation 9-3 of reference 8) **

$$\gamma = \frac{D \iint \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy}{- \iint \left[N'_x \left(\frac{\partial w}{\partial x} \right)^2 + N'_y \left(\frac{\partial w}{\partial y} \right)^2 + 2 N'_{xy} \left(\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right) \right] dx dy} \quad (2)$$

where γ is a common factor in N_x, N_y , and N_{xy} , so that

$$N_x = \gamma N'_x, \quad N_y = \gamma N'_y, \quad N_{xy} = \gamma N'_{xy}$$

* Both equations 1 and 2 assume that the membrane forces N_x, N_y , and N_{xy} are independent of the buckled deflection. This assumption is satisfied for the problem of incipient buckling.

** Pellett (7) derived this expression independently; however, he made an error which, together with another error (see page 24), invalidated his results.

are the membrane forces per unit length.

For the calculation of χ it is necessary to find an expression for w which satisfies the boundary conditions and minimizes χ , i.e., the variation of χ (in equation 2) must be zero.

Adopting the notation $w_r = \frac{\partial w}{\partial r}$, $w_\theta = \frac{\partial w}{\partial \theta}$, etc., and transforming this equation into cylindrical coordinates, with the proper limits of integration, results in the equation

$$I_1 + \chi I_2 = 0 \quad (3)^*$$

where
$$I_1 = D \int_0^{2\pi} \int_0^1 \left\{ \left[w_{rr} + \frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2} \right]^2 - (2-\nu) \left[w_{rr} \left(\frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2} \right) - \left(\frac{w_\theta}{r^2} - \frac{w_{r\theta}}{r} \right)^2 \right] \right\} r dr d\theta \quad (4)$$

and
$$I_2 = \int_0^{2\pi} \int_0^1 \left[N'_r w_r^2 + N'_\theta \frac{w_\theta^2}{r^2} + 2 N'_{r\theta} \frac{w_r w_\theta}{r} \right] r dr d\theta \quad (5)^*$$

The values of N'_r , N'_θ and $N'_{r\theta}$ which are to be used in equation 5 are obtained from Kirsch's solution (6) for the membrane stress distribution, namely,

$$\sigma_r = \frac{S}{2} \left[\left(1 - \frac{1}{r^2} \right) + \left(1 + \frac{3}{r^4} - \frac{4}{r^2} \right) \cos 2\theta \right] \quad (6)$$

$$\sigma_\theta = \frac{S}{2} \left[\left(1 + \frac{1}{r^2} \right) - \left(1 + \frac{3}{r^4} \right) \cos 2\theta \right] \quad (7)$$

$$\tau_{r\theta} = -\frac{S}{2} \left[1 - \frac{3}{r^4} + \frac{2}{r^2} \right] \sin 2\theta \quad (8)$$

*The equation (eq 5) which expresses I_2 in this paper is the negative of that which Timoshenko and Gere use. This accounts for equation (3) being slightly different than that given by Timoshenko and Gere.

where S is the applied tensile stress (see Figure 1).

The membrane forces may be expressed as $N_r = \sigma_r h$, $N_\theta = \sigma_\theta h$, and $N_{r\theta} = \tau_{r\theta} h$. It is now apparent that γ may be interpreted as S in the above equations. Using the notation $\sigma'_r = \frac{\sigma_r}{S}$, etc., equations 3 and 5 become respectively

$$I_1 + S I_2 = 0 \quad (9)$$

$$I_2 = h \int_0^{2\pi} \int_1^\infty \left[\sigma'_r w_r^2 + \sigma'_\theta \frac{w_\theta^2}{r^2} + 2 \tau'_{r\theta} \frac{w_r w_\theta}{r} \right] r dr d\theta \quad (10)$$

The assumed expression for w must have sufficient disposable parameters that the boundary conditions may be satisfied and that the actual lateral deflection may be closely approximated. Pellett (7) chose the following excellent expression for w

$$w(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} r^{-n} \left[1 + \frac{X_{nm}}{r} + \frac{Y_{nm}}{r^2} \right] \left\{ \begin{array}{c} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{array} \right\} \quad (11)$$

where A_{nm} and B_{nm} are arbitrary coefficients that are varied in order to minimize S in equation 9, coefficients X_{nm} and Y_{nm} having been chosen so as to satisfy the boundary conditions.

From Figure 1 it is seen that there is loading symmetry with respect to both the x and y axes. This means that when w is written as a Fourier series in θ (eq. 11) the series contains either cosine or sine terms, but not both. Additionally, the requirement of symmetry around both the x and y axes means that only odd or even terms in m need be used. Thus the Fourier portion of equation 11 reduces to:

$$(a) A_{nm} \cos m\theta, \quad m = 0, 2, 4, \dots \quad (\text{Sym A})^*$$

Symmetry with respect to the x and y axes

$$(b) \text{ or } A_{nm} \cos m\theta, \quad m = 1, 3, 5, \dots \quad (\text{Sym B})$$

Symmetry with respect to the x axis

Antisymmetry with respect to the y axis

$$(c) \text{ or } A_{nm} \sin m\theta, \quad m = 1, 3, 5, \dots \quad (\text{Sym C})$$

Symmetry with respect to the y axis

Antisymmetry with respect to the x axis

$$(d) \text{ or } A_{nm} \sin m\theta, \quad m = 0, 2, 4, \dots \quad (\text{Sym D})$$

Antisymmetry with respect to the x and y axes

This results in four separate expressions for w , one for each possible condition of symmetry. The full series for w could be used, but if this is done a large amount of computer storage would be wasted because in any solution three quarters of the coefficients, A_{nm} and B_{nm} , would be zero. It will later be shown that each of these expressions for w represents a different buckling mode.

The function w must also satisfy certain boundary conditions. The stress or strain distribution is that resulting from the Kirsch solution (boundary conditions for which are already satisfied) and the infinitesimal lateral deflections w .

The conditions which must be satisfied at $r = \infty$, namely

$$w = w_r = w_{rr} = 0 \quad (12)$$

* These symmetry designations are used here simply to distinguish between the different types of symmetry. The letters agree with ascending order of buckling stress, S_{cr} , obtained in the analysis which follows.

are obviously satisfied by the choice of w given by equation 11 since w and all its derivatives vanish for $r = \infty$.

The edge of the hole is free. Following Timoshenko and Woinowsky-Krieger (9) and using the cylindrical coordinate system established in Figure 1, the conditions for a free edge are

$$\left[Q_r - \frac{1}{r} \frac{\partial M_{rt}}{\partial \theta} \right]_{r=1} = 0 \quad (13)$$

$$\left[M_r \right]_{r=1} = 0 \quad (14)$$

While the work necessary to show that the expression for w , equation 11, satisfies these equations is not difficult, it is somewhat involved and therefore is shown separately in Appendix A. Formulas for the quantities Q_r , M_{rt} , and M_r are also given in Appendix A.

It is sufficient for our purposes here to say that unique values of X_{nm} and Y_{nm} are found which allow each term of equation 11 to satisfy the free edge boundary conditions for the values of n and m chosen.

The procedure is now straightforward even if very complicated and detailed. Inasmuch as Pellett's thesis contains the details (particularly his Appendices B and C which cannot be abbreviated without losing intelligibility) this material will not be repeated here. The reader who wishes to examine the details more closely is referred to Pellett's thesis (7). Briefly the procedure is as follows. The Kirsch solution, equations 6 — 8, with the stress S factored out, is substituted into equation 10. The appropriate partial derivatives of w are formed from equation 11 and substituted into equations 4 and 10. Note that in these integrals the derivatives of w appear to the second degree. Since w itself and each of its derivatives is represented as a double series, such second degree terms are quadruple series. Evidently there is considerable manipulation to be done and a great deal of "book-keeping" which must be managed.

The actual integrations are elementary although Pellett finds it convenient to devote an entire appendix (his Appendix C) to some of the details. The upshot is that after the substitutions are made, the multiplications performed, and the integrations carried out, the energy integrals I_1 and I_2 take the form

$$I_1 = D \sum_I \sum_J \sum_K \sum_L P_{IJKL} A_{IJ} A_{KL} \quad (15)*$$

$$I_2 = h \sum_I \sum_J \sum_K \sum_L Q_{IJKL} A_{IJ} A_{KL} \quad (16)$$

where P_{IJKL} and Q_{IJKL} are numerical coefficients. Actually, as may be seen from the original forms for I_1 and I_2 , they are really functions of Poisson's ratio, ν , but because of evident difficulties no attempt has been made to retain the literal dependence upon ν ; rather, the evaluations which are actually performed incorporate a numerical value for ν at an early stage. Most of the calculations were actually carried out with a value of $\nu = 0.3$. However, supplementary calculations were performed using other values of ν so as to assess the dependence of the final results upon the choice of ν .

In equations 15 and 16, the quantities A_{IJ} and A_{KL} represent the double series of as-yet unknown coefficients in equation 11. It does not seem at all possible to carry out the remainder of the evaluation in literal terms so that recourse is made to a digital computer evaluation. Obviously, then, it is necessary to truncate the series given in equation 11. Thus we take the summations in equation 11 so as to include N terms, M terms

*The unknown coefficients for the cosine series, A_{nm} , only will be used henceforth, although these could be the sine series coefficients, B_{nm} equally as well.

respectively. Later it will be seen that different choices of N and M lead to slightly different values for the critical stress, and advantage will be taken of this fact to extrapolate the results so as to obtain a better evaluation than would otherwise be possible within the limits of computer storage and time availability.

With such a truncation, it is clear that the number of unknown quantities A_{nm} is $\bar{J} = N \cdot M$, and it is convenient to order these quantities into a single, singly subscripted array, which we do as follows. We write

$$A_{I'} = A_{IJ}$$

where

$$I' = (I - 1)M + J + 1$$

Using the same device, the quadruply subscripted quantities P_{IJKL} and Q_{IJKL} are written as doubly subscripted arrays $P_{I',J'}$ and $Q_{I',J'}$, where each of the subscripts runs from 1 to $\bar{J}$.

To make this operation clearer, the following example is offered. Suppose that the value $M = 4$ has been chosen (the value of N does not matter in this renumbering) and we wish to convert $Q_{5432} = Q_{(5,4)(3,2)}$ to a doubly subscripted quantity. We have $I = 5$, $J = 4$, $K = 3$, and $L = 2$. From the formula above, we calculate $I' = 21$, $J' = 11$. Thus we write the quadruply subscripted quantity Q_{5432} (i.e., que sub five four three two) as $Q_{21,11}$ (i.e., que sub twenty-one eleven). With this change in notation, equations 15 and 16 may be written as

$$I_1 = D \sum_{I'=1}^{\bar{J}} \sum_{J'=1}^{\bar{J}} P_{I',J'} A_{I'} A_{J'} \quad (17)$$

$$I_2 = h \sum_{I'=1}^{\bar{J}} \sum_{J'=1}^{\bar{J}} Q_{I',J'} A_{I'} A_{J'} \quad (18)$$

A further simplification results if matrix notation is adopted at

this time. In matrix form equations 17 and 18 become

$$I_1 = D \bar{A}^T \underline{P} \bar{A} \quad ; \quad I_2 = h \bar{A}^T \underline{Q} \bar{A} \quad (19a,b)$$

where

$$\bar{A} = \text{col} \{ A_1, \dots, A_J \}$$

and $\underline{P}$, $\underline{Q}$ are symmetrical matrices of order $(J \times J)$. The unknown A_i 's must be such that S is stationary (a minimum) for arbitrary variations in the A_i 's. Thus, defining

$$\delta \bar{A} = \text{col} \{ \delta A_1, \dots, \delta A_J \} \quad (20)$$

$$\frac{\partial}{\partial \bar{A}} = \left[\frac{\partial}{\partial A_1}, \dots, \frac{\partial}{\partial A_J} \right] \quad (21)$$

we proceed as follows*

$$\begin{aligned} \delta S &= \frac{\partial S}{\partial \bar{A}} \delta \bar{A} = \frac{\partial}{\partial \bar{A}} \left[-\frac{I_1}{I_2} \right] \delta \bar{A} \\ &= \frac{1}{I_2} \left[\frac{\partial I_1}{\partial \bar{A}} + S \frac{\partial I_2}{\partial \bar{A}} \right] \delta \bar{A} \end{aligned} \quad (22)$$

Using equation 19 and the analysis in reference (5)

$$\delta S = \frac{2D}{I_2} \left[\frac{Sh}{D} \underline{Q} + \underline{P} \right] \bar{A} \delta \bar{A} \quad (23)$$

*The second error in Pellett's work occurred when he differentiated incorrectly in the process of obtaining the variation of S .

Since δS is to be zero for arbitrary δA , we are led to

$$2 \left[\frac{sh}{D} \underline{Q} + \underline{P} \right] \underline{A} = 0 \quad (24)$$

When the substitution $\lambda = \frac{sh}{D} = \frac{12S(1-\nu^2)}{E h^2}$ is made, the following matrix eigenvalue problem results

$$[\lambda \underline{Q} + \underline{P}] \underline{A} = 0 \quad (25)$$

The method used to solve this matrix eigenvalue problem is shown in Appendix D. There are as many solutions, eigenvalues λ , as there are elements of $\underline{A}$. The smallest positive eigenvalue, which corresponds to the minimum applied tensile stress, is of the most interest. Larger positive values of λ correspond to more complex buckled mode shapes while the negative values correspond to buckled modes due to applied compressive stress.

The value of the buckling stress is recovered from the definition of λ . That is, we have

$$S_{cr} = [\lambda / 12 (1 - \nu^2)] E h^2$$

However, we recognize, from consideration of the dimensions involved, that we can restore the analysis to a case of hole radius other than unity simply by writing t/a in place of h . Also, we denote the term in square brackets by the symbol K , which we call the buckling coefficient. Note that the effect of varying Poisson's ratio cannot be inferred from the expression in square brackets inasmuch as Poisson's ratio has been intimately involved in the entire analysis, starting with equations 1 and 2.

To investigate the effect of varying Poisson's ratio it is necessary to recalculate the quantities P_{IJKL} and Q_{IJKL} using different values of ν^* and also to vary ν in the expression $(1 - \nu^2)$.

Thus we finally arrive at the expression

$$S_{cr} = K E (t/a)^2 \quad (26)$$

where K is a function only of ν . Different modes, that is, different values of the eigenvalues λ , and different assumed conditions of symmetry, lead to different evaluations for K , of which, naturally, the smallest is of greatest interest by far.

*Different values of ν can be introduced in the computations by changing card 0011 in the main program and card 0003 in the subroutine BOUND; see computer program listing, Appendix E.

3. Results.

Computed values of K in equation 26 are shown in Table 2; these values assume $\nu = 0.3$. N and M in Table 2 are the number of terms taken in n and m in the assumed deflection, equation 11. As both N and M are increased, the value of K appears to approach a limit. In Figures 2 and 3, K is plotted versus M for varying values of N . In Figure 4, K is plotted versus N for varying values of M . The trend toward a limit is more obviously shown in these curves.

Because of limitations on computer storage space and time, $N = 8$ and $M = 10$ are the practical upper limits for the number of terms that can be used in the series expression for deflection.

Similar curves and tables could be exhibited for other values of ν . However, since they are similar to those for $\nu = 0.3$, they are omitted.

A tabulation of K for $N = 8$ and $M = 10$ is shown in Table 1 below.

Table 1

Values of the Buckling Coefficient, K , for $N = 8$
and $M = 10$

Mode	Sym.*	$\nu = 0.0$	$\nu = 0.2$	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.5$
1	A	1.786	1.723	1.726	1.752	1.808
2	B	2.106	2.026	2.025	2.051	2.109
3	C			3.332		
4	D			3.651		
5	A			5.646		
6	B			6.377		
7	C			7.851		

*These symmetry types correspond to the symmetries discussed on page 20.

Table 2

Values of the Buckling Coefficient, K, for Various Values of M and N ($\nu = 0.3$)

Mode 1

		M = 6	7	8	9	10	11	12	13
N = 6		1.7450	-	1.7311	1.7286	1.7274	1.7269	1.7267	1.7266
7		-	1.7352	1.7305	1.7277	1.7262	1.7254		
8		-	1.7348	1.7302	1.7274	1.7257			

Mode 2

		M = 6	7	8	9	10
N = 6		2.0291	-	-	-	-
7		-	2.0260	2.0255	2.0254	2.0253
8		-	2.0257	2.0252	2.0250	2.0249
9		-	2.0255			

Mode 3

		M = 6	7	8	9	10
N = 6		3.3438	-	-	-	-
7		-	3.3349	3.3331	3.3325	3.3323
8		-	3.3348	3.3329	3.3324	3.3321

Mode 4

		M = 6	7	8	9	10
N = 6		3.6587	-	-	-	-
7		-	3.6529	3.6524	3.6522	3.6521
8		-	3.6518	3.6513	3.6512	3.6511

(no values were calculated where blanks are indicated)

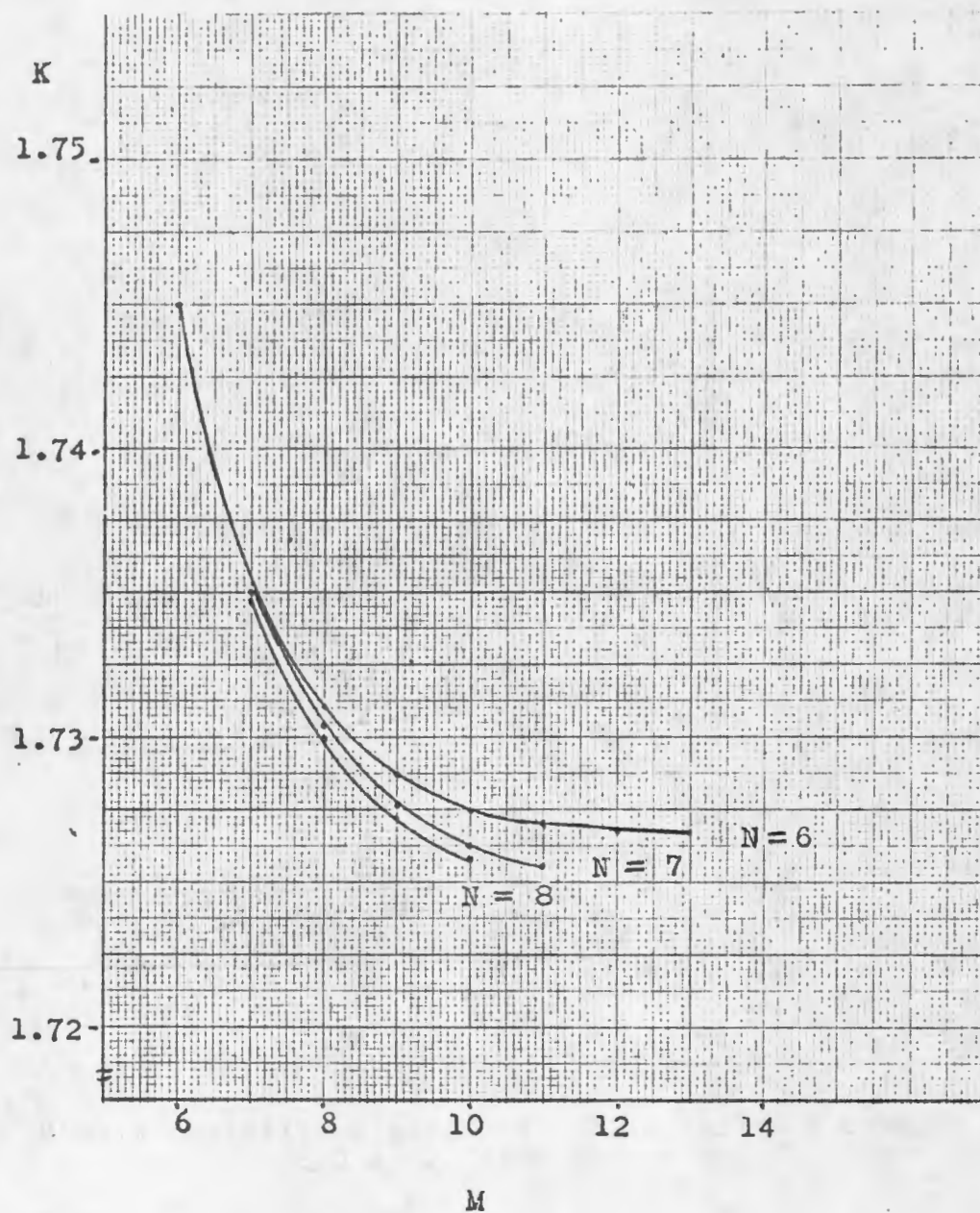


Figure 2 - Plot of the Buckling Coefficient K vs M
for mode 1 with $\nu = 0.3$

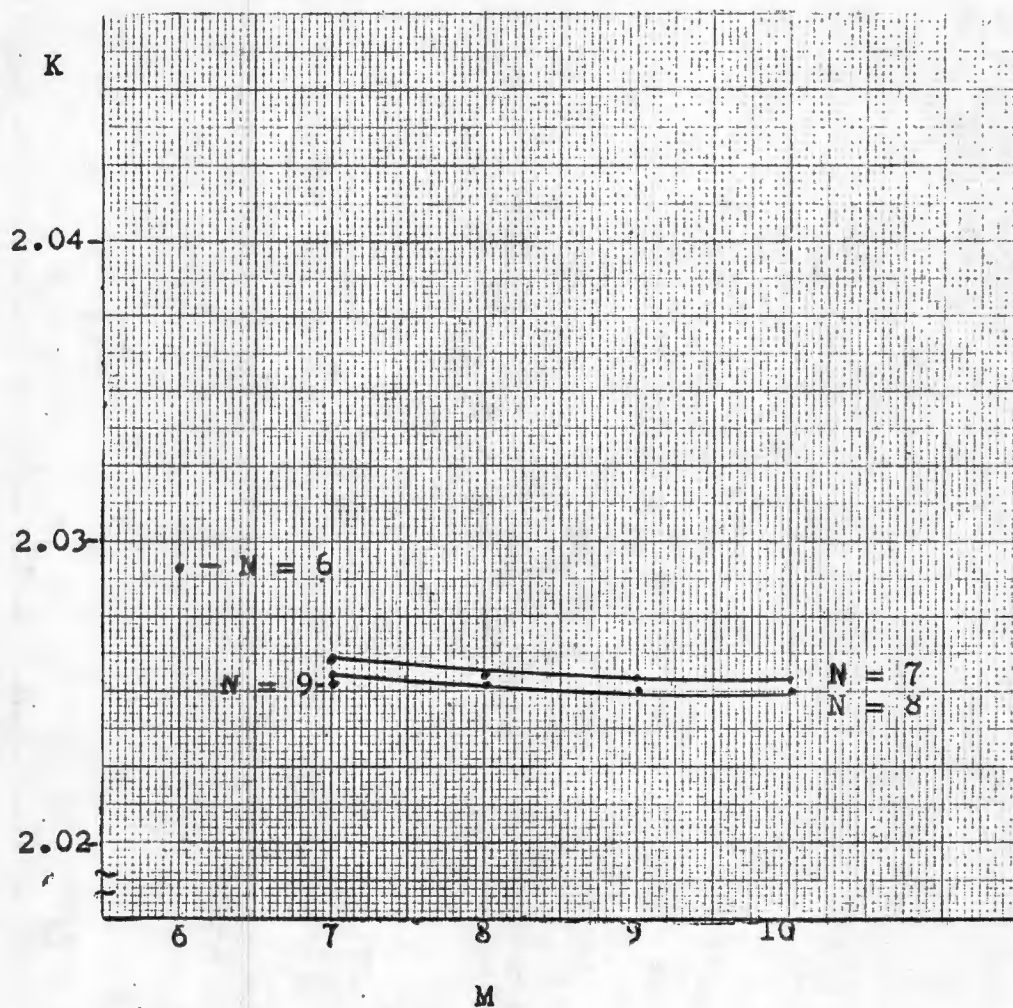


Figure 3 - Plot of the Buckling Coefficient K vs M
for mode 2 with $\nu = 0.3$

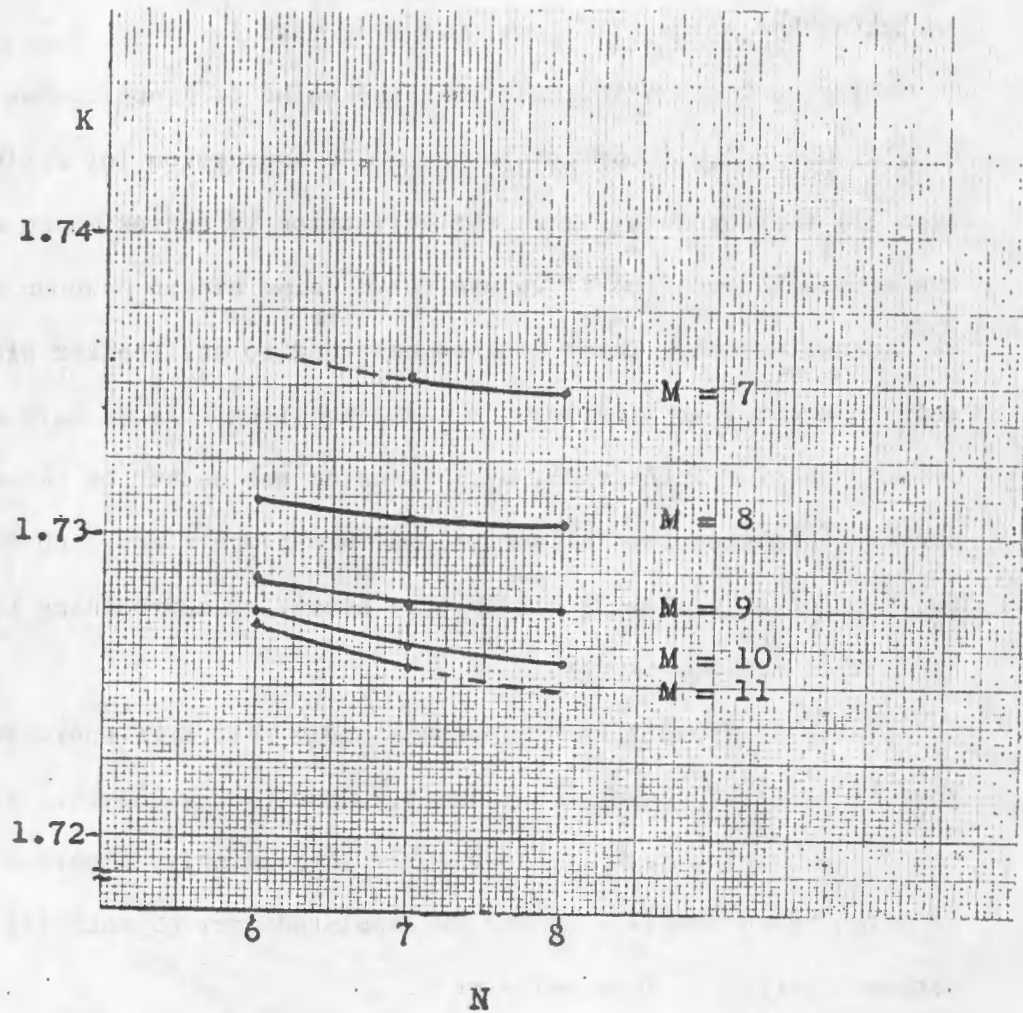


Figure 4 - Plot of the Buckling Coefficient K vs N for mode 1 with $\nu = 0.3$

Since the lowest value of K and corresponding critical stress is of primary interest and $\nu = 0.3$ is a common value, more time was devoted analyzing the value of K for $\nu = 0.3$, mode 1.

Figures 2 and 4 indicate that the value of K approaches a lower limit or minimum as the number of terms in the expression for deflection (equation 11) is increased. When the deflection is approximated by truncating the series in equation 11 we are introducing hidden unknown constraints on the deflection. These constraints tend to stiffen the plate and require a greater applied load to cause buckling. As we more nearly approach the true deflection, by increasing the number of terms used in the series expression for deflection, we more closely approach the value of K for the actual plate. It is in this sense, of approaching the true value of K , that minimum is used.

While it appears from Figures 2 and 4 that K is approaching a limiting value, it is not apparent exactly what that value is. It appears that the minimum value of K is about 1.72. Accordingly, a method was sought by which these curves could be extrapolated very objectively in order to obtain a better minimum value of K .

Using a least squares method* of curve fitting the minimum value of K was determined to be 1.720.

For any symmetry type such as Sym A (symmetric with respect to both the x and y axes) there are several** values of K which correspond to

* See Appendix C for the details involved in this technique.

** An infinite number of such values would be expected if it had been possible to use an infinite number of unknown coefficients A_{nm} ; however with the use of truncated series there are only a finite number of such values.

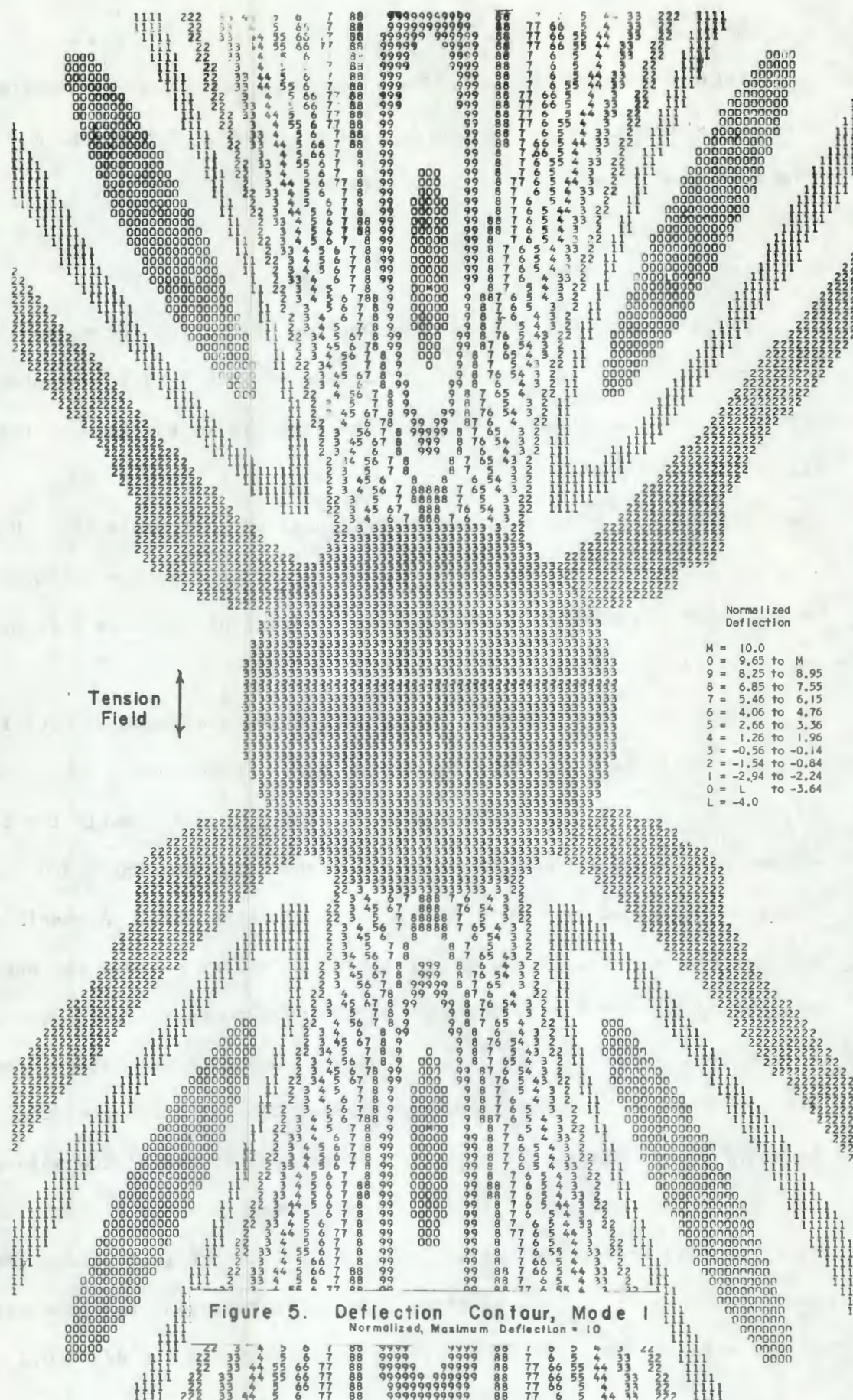
increasingly complex buckled deflections, each of which retains the same symmetry. For example, for Sym A., with $\nu = 0.3$, $N = 8$, and $M = 10$, the following values of K were determined.

Mode =	1	5	9	12
K =	1.726	5.646	9.205	12.086

Figures 5 and 6 illustrate the buckled deflection for modes 1 and 2 respectively. The more complex deflection of mode 5 is illustrated in Figure 7. It can also be seen that the symmetry is the same as for mode 1.

Figure 8 is taken from the experimental work of Danis (2). His results for finite plates are plotted as K versus d/b , the ratio of hole diameter to plate width. The buckling coefficient for mode 1 is plotted on the y axis ($d/b = 0$).

There appears to be some agreement between the value of buckling coefficient calculated in this thesis and Danis' experimental work. The minimum value of K calculated was 1.720 (for $\nu = 0.3$), while the lower values that Danis found were between 1.5 and 2.5. For small d/b ratios Danis found values of K that were much higher than this. A possible explanation may be that the strain gages he used to measure the buckling reinforced the areas near the edge of the hole so as to resist buckling. If the area covered by the strain gage was to become greater in proportion to the area that was likely to buckle, the reinforcing effect would become greater. This is exactly the situation for Danis' low values of d/b , as he used smaller holes in the same size plates to vary the ratio of d/b . This reinforcing effect would raise the critical stress required to cause buckling. Also, it seems reasonable to assume that the effect of the edge of the plate would not be much different for $d/b = 0.3$ or



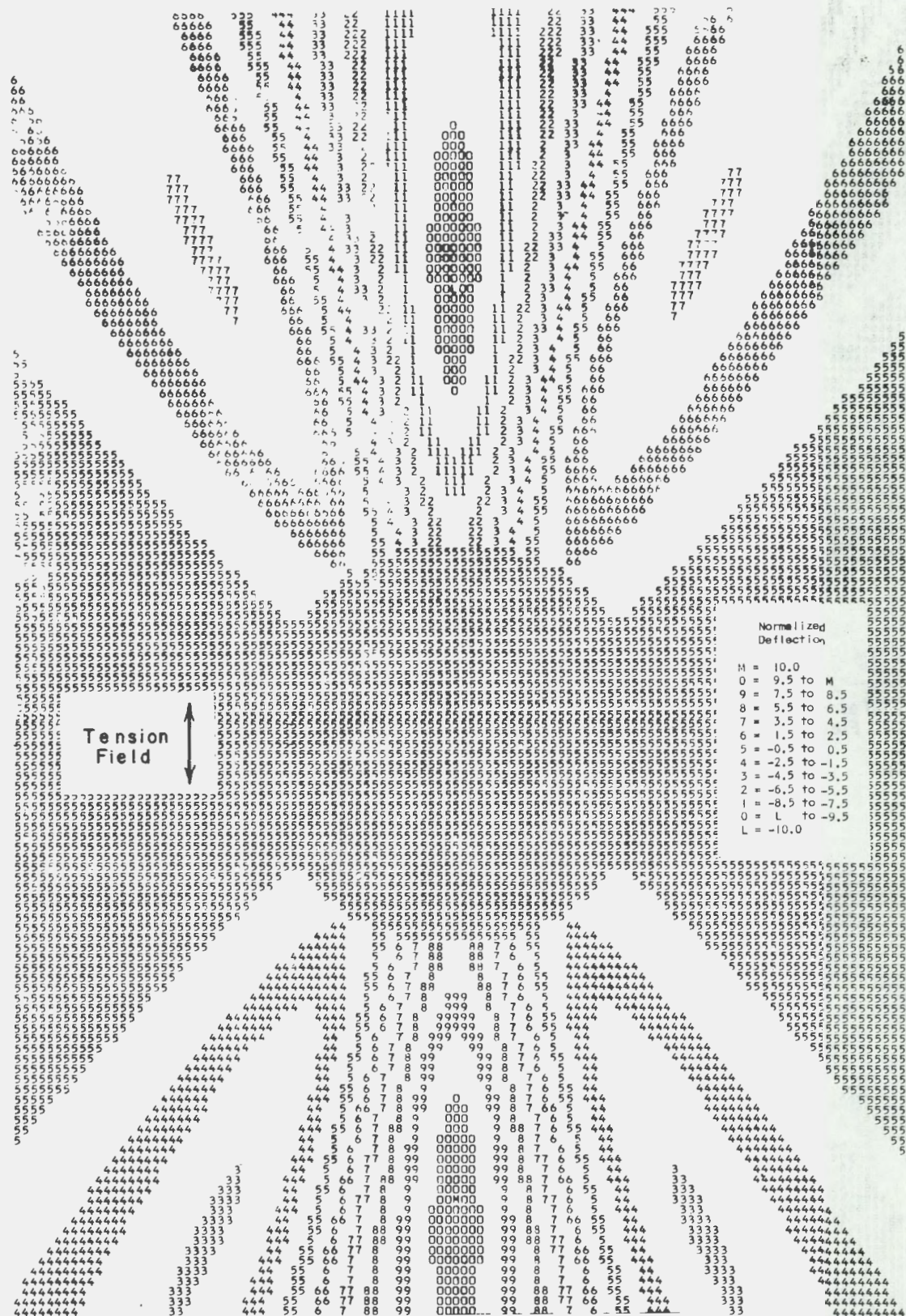
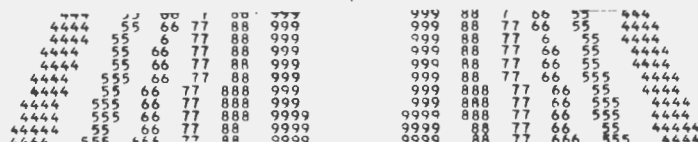
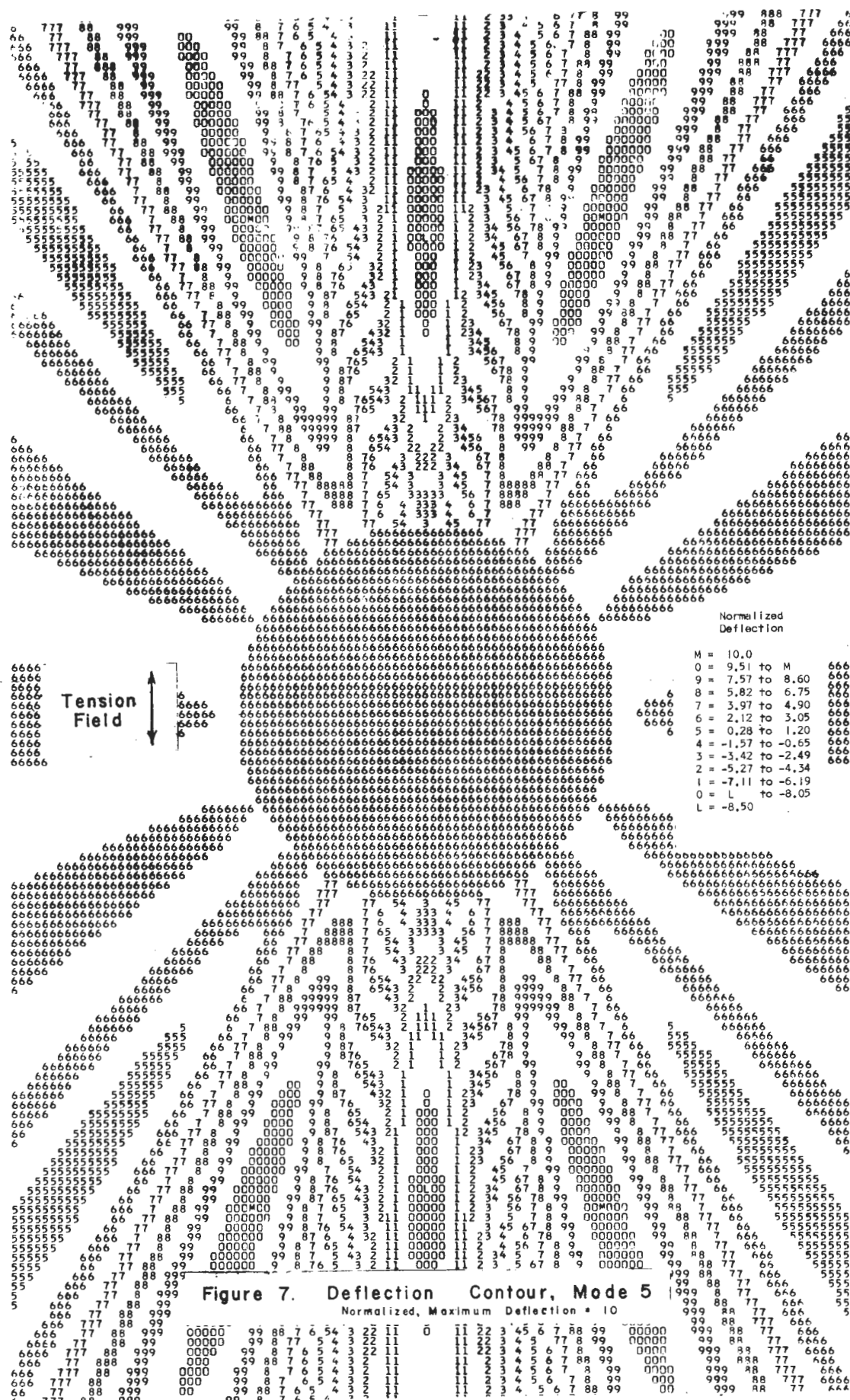


Figure 6. Deflection Contour, Mode 2

Normalized, Maximum Deflection = 10





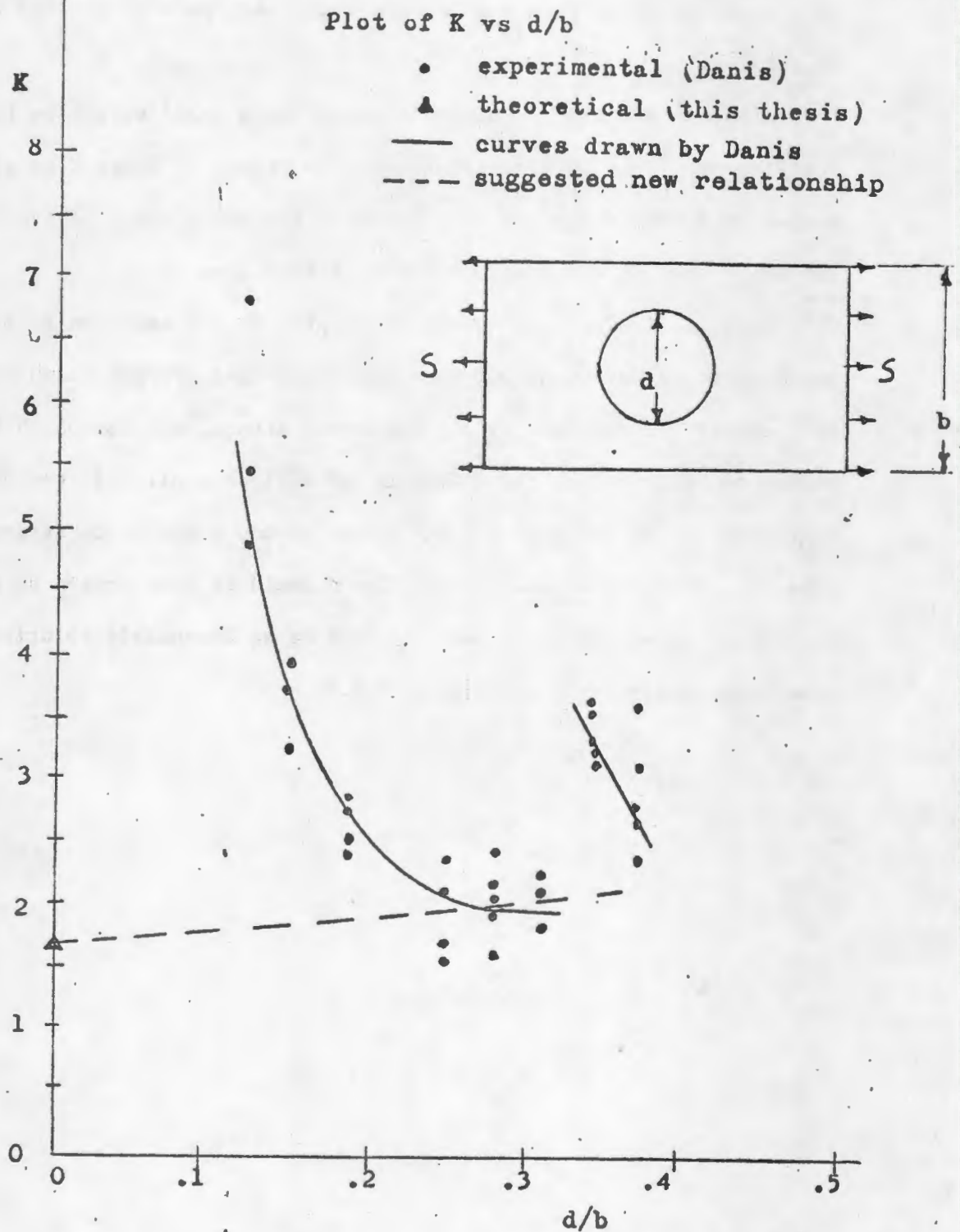


Figure 8 - Comparison with Experimental Data of Danis

$d/b = 0.1$. In Figure 8 the solid curves are Danis's and the dashed curve is a postulation of what the results should be, on the basis of the above discussion.

As seen in Table 1, Poisson's ratio has a small effect on the buckling coefficient. This is also illustrated in Figure 9, where K is plotted versus Poisson's ratio for mode 1 with $N = 8$ and $M = 10$. The difference between K for $\nu = 0.3$ and $\nu = 0.0$ is less than 4%.

This small effect of ν may be helpful in the solution of other buckling problems. Consider some other buckling problem which has not been successfully solved by any method of attack, but for which there is reason to believe that the effect of ν will be small. If the problem is simplified by letting $\nu = 0$, it may become amenable to attack, perhaps by a theoretical analysis. There would be good reason to conjecture that the solution for $\nu = 0$ would be an acceptable solution for practical values of ν such as $\nu = 0.3$.

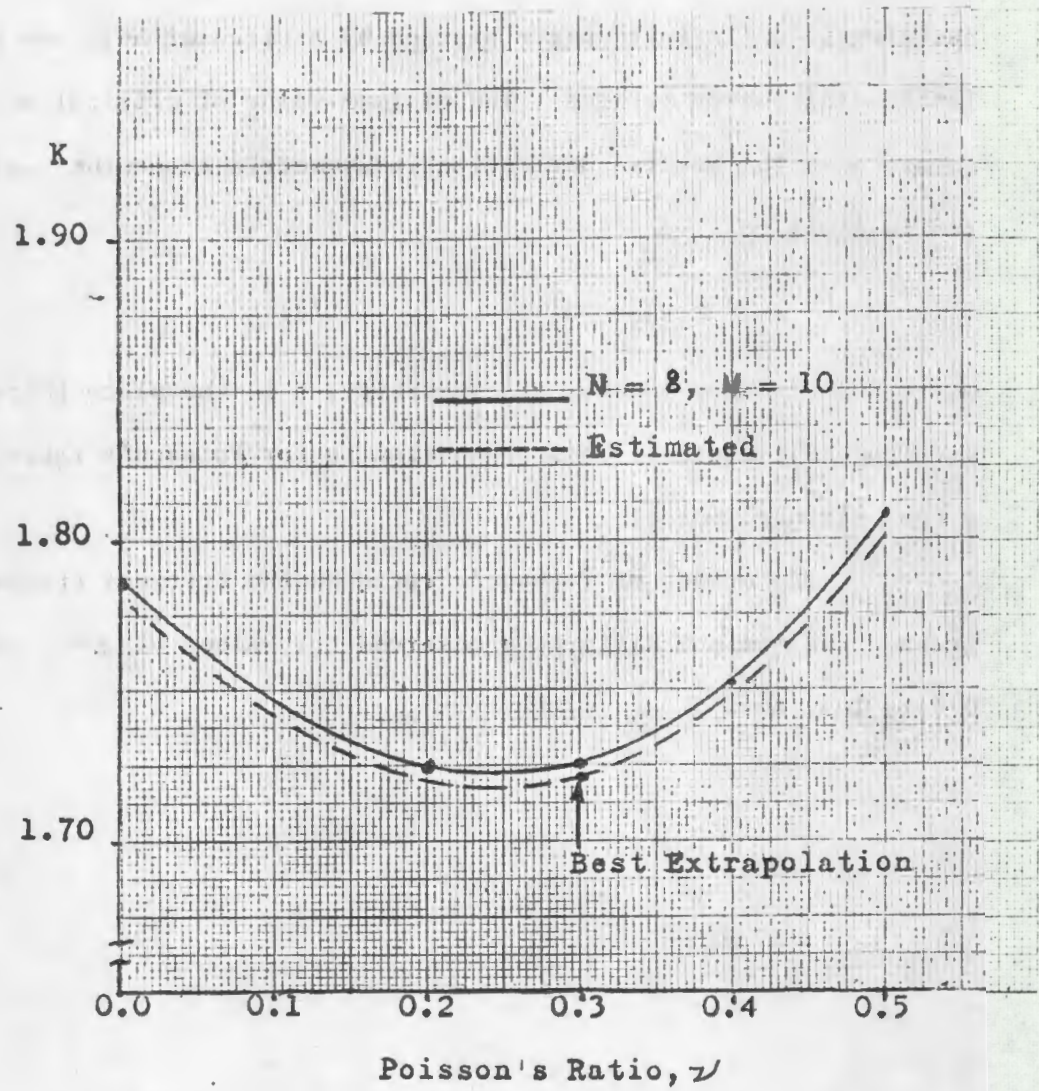


Figure 9 - Plot of the Buckling Coefficient K vs Poisson's Ratio ν .

4. Conclusions.

The critical applied tensile stress that will cause incipient buckling in an infinite plate pierced by a circular hole can be calculated using energy methods. The minimum value of critical stress, S_{cr} , occurs when the buckled deflection is symmetric about the x and y axes and is equal to

$$S_{cr} = 1.720 E(t/a)^2$$

where E is Young's modulus of elasticity, t is the plate thickness, and a is the hole radius. This evaluation is for Poisson's ratio $\nu = 0.3$, a commonly used value.

The effect of Poisson's ratio on the critical stress is not great. The maximum difference observed for values of ν from 0 to 0.5 is less than 5%.

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APPENDIX A*

Satisfaction of the Boundary Conditions

The free edge boundary conditions for the hole are stated in equations 13 and 14 as

$$\left(Q_r - \frac{1}{r} \frac{\partial M_{rt}}{\partial \theta} \right)_{r=1} = 0 \quad (\text{A-1})$$

$$(M_r)_{r=1} = 0 \quad (\text{A-2})$$

Timoshenko and Woinowsky-Krieger (9) show the expressions for Q_r , M_{rt} , and M_r to be

$$Q_r = -D \frac{\partial}{\partial r} \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] \quad (\text{A-3})$$

$$M_r = -D \left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \quad (\text{A-4})$$

$$M_{rt} = D(1-\nu) \left[\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right] \quad (\text{A-5})$$

Adopting the notation $\frac{\partial w}{\partial r} = w_r$, etc., and performing the appropriate differentiations and substitutions, the free edge boundary conditions become

$$\left[w_{rrr} + \frac{w_{rr}}{r} - \frac{w_r}{r^2} + \frac{(2-\nu)}{r^2} w_{r\theta\theta} - \frac{(3-\nu)}{r^3} w_{\theta\theta} \right]_{r=1} = 0 \quad (\text{A-6})$$

$$\left[w_{rr} + \frac{\nu}{r} (w_r + \frac{w_{\theta\theta}}{r}) \right]_{r=1} = 0 \quad (\text{A-7})$$

* Appendices A and B are essentially the work of Pellett (1). Only minor changes in notation have been made to correspond to changes in the main text.

The lateral plate deflection w , as expressed in equation 11 is

$$w(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} r^{-n} \left[1 + \frac{X_{nm}}{r} + \frac{Y_{nm}}{r^2} \right] \left\{ \begin{array}{c} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{array} \right\} \quad (\text{A-8})$$

where the coefficients A_{nm} and B_{nm} are varied to minimize the plate energy, and the coefficients X_{nm} and Y_{nm} are chosen so that the boundary conditions expressed by equations A-6 and A-7 are satisfied.

The derivatives of w are tabulated in Table 3. When these derivatives are substituted into equations A-6 and A-7, and evaluated at $r = 1$, the following equations result

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (s_0 + s_1 X_{nm} + s_2 Y_{nm}) \left[\begin{array}{c} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{array} \right] = 0 \quad (\text{A-9})$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (t_0 + t_1 X_{nm} + t_2 Y_{nm}) \left[\begin{array}{c} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{array} \right] = 0 \quad (\text{A-10})$$

where

$$s_i = (n+i)(n+i+1-\nu) - m^2 \nu \quad (\text{A-11})$$

$$t_i = (3-\nu)m^2 + (2-\nu)m^2(n+i) - (n+i)^2(n+i+2) \quad (\text{A-12})$$

for $i = 0, 1, 2$

Equations A-9 and A-10 must hold for all values of θ , which can be true only if

$$s_0 + s_1 X_{nm} + s_2 Y_{nm} = 0 \quad (\text{A-13})$$

$$t_0 + t_1 X_{nm} + t_2 Y_{nm} = 0 \quad (\text{A-14})$$

for any value of n and m . Thus, a value of X_{nm} and Y_{nm} can be found for each value of n and m , such that the boundary conditions are satisfied.

The results of solving equations A-13 and A-14 simultaneously are

$$X_{nm} = (s_2 t_0 - s_0 t_2) / (s_1 t_2 - s_2 t_1) \quad (A-15)$$

$$Y_{nm} = (s_0 t_1 - s_1 t_0) / (s_1 t_2 - s_2 t_1) \quad (A-16)$$

The above results are incorporated into SUBROUTINE BOUND (see Appendix E) which generates values of X_{nm} and Y_{nm} for given values of n and m .

$$\begin{aligned}
w &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[r^{-n} + X_{nm} r^{-(n+1)} + Y_{nm} r^{-(n+2)} \right] \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_r &= -\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[n r^{-(n+1)} + (n+1) X_{nm} r^{-(n+2)} + (n+2) Y_{nm} r^{-(n+3)} \right] \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_{rr} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[n(n+1) r^{-(n+2)} + (n+1)(n+2) X_{nm} r^{-(n+3)} + (n+2)(n+3) Y_{nm} r^{-(n+4)} \right] \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_{rrr} &= -\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[n(n+1)(n+2) r^{-(n+3)} + (n+1)(n+2)(n+3) X_{nm} r^{-(n+4)} + (n+2)(n+3)(n+4) Y_{nm} r^{-(n+5)} \right] \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_{re} &= -\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[n r^{-(n+1)} + (n+1) X_{nm} r^{-(n+2)} + (n+2) Y_{nm} r^{-(n+3)} \right] m \begin{cases} -A_{nm} \sin m\theta \\ \text{or} \\ B_{nm} \cos m\theta \end{cases} \\
w_{ree} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[n r^{-(n+1)} + (n+1) X_{nm} r^{-(n+2)} + (n+2) Y_{nm} r^{-(n+3)} \right] m^2 \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_{\theta\theta} &= -\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[r^{-n} + X_{nm} r^{-(n+1)} + Y_{nm} r^{-(n+2)} \right] m^2 \begin{cases} A_{nm} \cos m\theta \\ \text{or} \\ B_{nm} \sin m\theta \end{cases} \\
w_{\theta} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[r^{-n} + X_{nm} r^{-(n+1)} + Y_{nm} r^{-(n+2)} \right] m \begin{cases} -A_{nm} \sin m\theta \\ \text{or} \\ B_{nm} \cos m\theta \end{cases}
\end{aligned}$$

Table 3

Tabulation of the Derivatives of w

APPENDIX B*

Solution of the Eigenvalue Problem $(\underline{Q} \lambda + \underline{P}) \underline{A} = 0$

The canonical form of the matrix eigenvalue problem is

$$(\underline{I} x - \underline{B}) \underline{X} = 0 \quad (B-1)$$

where $\underline{I}$ is the identity matrix, $\underline{X}$ an unknown eigenvector, x is an unknown scalar parameter (eigenvalue), and $\underline{B}$ a known square coefficient matrix.

Equation B-1 has a nontrivial solution if and only if the determinant

$$\det(\underline{I} x - \underline{B}) = 0 \quad (B-2)$$

If this determinant is expanded, a polynomial equation in x results. However, if the determinant is of order greater than three or four, it is generally inefficient to obtain this polynomial equation and then solve it. In the actual numerical calculation in this thesis, the so-called Jacobi method (3) was employed.

Consider the eigenvalue problem (equation 25)

$$(\underline{Q} \lambda + \underline{P}) \underline{A} = 0 \quad (B-3)$$

where both Q and P are symmetric matrices. Dividing equation B-3 by λ and making the substitution

$$y = - \frac{1}{\lambda} \quad (B-4)$$

gives

$$(\underline{P} y - \underline{Q}) \underline{A} = 0 \quad (B-5)$$

Any symmetric matrix with real elements can be reduced to a product of the eigenvector and eigenvalue matrices, so that

* See footnote page 42.

$$\underline{\underline{P}} = \underline{\underline{R}} \underline{\underline{D}} \underline{\underline{R}}^T \quad (B-6)$$

where $\underline{\underline{D}}$ is a diagonal matrix of the eigenvalues (d_i) of $\underline{\underline{P}}$, and $\underline{\underline{R}}$ is a matrix of normalized eigenvectors ($\underline{r}_i$) obtained from a solution of the problem

$$(\underline{\underline{I}} \, d_i - \underline{\underline{P}}) \underline{r}_i = 0 \quad (B-7)$$

Note that $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$ since the matrix $\underline{\underline{P}}$ is symmetric and has orthogonal eigenvectors.

Equation B-5 may now be written

$$(\underline{\underline{R}} \underline{\underline{D}} \underline{\underline{R}}^T \underline{y} - \underline{\underline{Q}}) \underline{A} = 0 \quad (B-8)$$

Premultiplying by $\underline{\underline{R}}^T$, we get

$$(\underline{\underline{D}} \underline{y} - \underline{\underline{R}}^T \underline{\underline{Q}}) \underline{A} = 0 \quad (B-9)$$

Factoring $\underline{\underline{R}}$ out of $\underline{A} = \underline{\underline{I}} \underline{A} = \underline{\underline{R}} \underline{\underline{R}}^T \underline{A}$, and letting

$$\underline{y} = \underline{\underline{R}}^T \underline{A} \quad (B-10)$$

the problem reduces to

$$(\underline{\underline{D}} \underline{y} - \underline{\underline{R}}^T \underline{\underline{Q}} \underline{\underline{R}}) \underline{y} = 0 \quad (B-11)$$

Since $\underline{\underline{P}}$ is a result of the bending energy integral it can be expected to be positive definite*. $\underline{\underline{D}}$, the matrix of eigenvalues of $\underline{\underline{P}}$, therefore contains all positive terms and the square roots of its elements are real numbers; (only the positive square roots will be used). Premultiplying by $\underline{\underline{D}}^{-\frac{1}{2}}$, factoring, and making the substitution

$$\underline{z} = \underline{\underline{D}}^{\frac{1}{2}} \underline{y} \quad (B-12)$$

equation B-11 reduces to

$$(\underline{\underline{I}} \underline{y} - \underline{\underline{D}}^{-\frac{1}{2}} \underline{\underline{R}}^T \underline{\underline{Q}} \underline{\underline{R}} \underline{\underline{D}}^{-\frac{1}{2}}) \underline{z} = 0 \quad (B-13)$$

* During the numerical computations it was found that $\underline{\underline{P}}$ is in fact positive definite.

and becomes

$$(\underline{I}y - \underline{B}) \underline{Z} = 0 \quad (B-14)$$

which is of canonical form, where

$$\underline{B} = \underline{D}^{-\frac{1}{2}} \underline{R}^T \underline{Q} \underline{R} \underline{D}^{-\frac{1}{2}} \quad (B-15)$$

Note that $\underline{D}^{\frac{1}{2}}$ and $\underline{D}^{-\frac{1}{2}}$ are easily obtained by taking the square roots, and the square root reciprocals, respectively, of the diagonal terms. Also note that $\underline{B}$ is symmetric. The problem is now well suited for solution by use of the Jacobi method. It is seen that inaccuracies in obtaining eigenvectors $\underline{R}$ would cause compounded errors in the final solution for $\underline{A}$ and λ .

The problem is returned to original form with the substitutions

$$\lambda = -\frac{1}{y} \quad (B-16)$$

and

$$\underline{A} = \underline{R} \underline{D}^{-\frac{1}{2}} \underline{Z} \quad (B-17)$$

APPENDIX C

Least Squares Extrapolation of K

Since it is desirable to estimate the minimum buckling coefficient, the following method of extrapolation was used.

It is assumed that the buckling coefficient K is a function of the number of terms used in the series for w. Such a function may be

$$K = Y_1 + \frac{Y_2}{M^2} + \frac{Y_3}{M^4} + \frac{Y_4}{N^2} + \frac{Y_5}{N^4} + \frac{Y_6}{N^2 M^2} \quad (C-1)$$

where K is a calculated buckling coefficient, N and M are the number of terms in the series for deflection, and $Y_1, Y_2, \dots, Y_6$ are unknown coefficients.

If the coefficients of equation C-1 can be determined then Y_1 will be an estimate of the value of K as both N and M approach ∞ .

For a number of calculated values of K and known values of N and M a series of equations of the above form could be written in matrix form as follows

$$\underline{C} \cdot \underline{Y} = \underline{K} \quad (C-2)$$

where

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & & \\ C_{21} & C_{22} & & \\ & & \ddots & \\ & & & C_{Z6} \end{bmatrix} \quad \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_6 \end{bmatrix} \quad \underline{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_Z \end{bmatrix}$$

Z is the number of values of K that were calculated

$$C_{11} = 1, \quad C_{12} = 1/M_1^2, \text{ etc.}$$

A common method of fitting an equation to data points is the so-called method of least squares. In this method the sum of the squares of the errors, resulting from the use of a proposed equation such as equation

C-1, is minimized. That is, the variation of the sum of the squares of the errors with respect to the unknown coefficients, in this case the Y_1 , is set equal to zero.

Using ϵ to denote the sum of the squares of the errors, the resulting equation is

$$\epsilon = (\underline{C} \underline{Y} - \underline{K})^T (\underline{C} \underline{Y} - \underline{K}) \quad (C-3)$$

This equation may be expanded to the form

$$\epsilon = \underline{Y}^T \underline{C}^T \underline{C} \underline{Y} - 2 \underline{K}^T \underline{C} \underline{Y} + \underline{K}^T \underline{K} \quad (C-4)$$

If $\underline{Y}$ has variation $\delta \underline{Y}$, ϵ will have variation*

$$\delta \epsilon = \frac{\partial \epsilon}{\partial \underline{Y}} \delta \underline{Y} = 2(\underline{Y}^T \underline{C}^T \underline{C} - \underline{K}^T \underline{C}) \delta \underline{Y} \quad (C-5)$$

and for this to vanish for arbitrary variation in $\underline{Y}$, it is sufficient that

$$\underline{Y}^T = \underline{K}^T \underline{C} (\underline{C}^T \underline{C})^{-1} \quad (C-6)$$

It is easy to see that this gives a minimum since

$$\frac{\partial^2 \epsilon}{\partial \underline{Y}^2} = 2 \underline{C}^T \underline{C} \quad (C-7)$$

A digital computer was used to evaluate equation C-6 for the 15 values of K shown in Table 1, for mode 1, and $z' = 0.3$. The computer program used is shown in Appendix E. The results obtained are shown below

$$\begin{aligned} Y_1 &= 1.720 \\ Y_2 &= 0.1479 \\ Y_3 &= 30.93 \\ Y_4 &= 0.1202 \\ Y_5 &= 3.553 \\ Y_6 &= 12.17 \end{aligned}$$

* Differentiation of expressions of this type is discussed by Frazer, Duncan and Collar (4),

Other functions expressing the dependence of K upon N and M could be assumed, such as using odd negative exponents for both N and M . When this was done the coefficients associated with the odd negative exponents were found to be either zero or much smaller in magnitude than the coefficients listed above.

APPENDIX D

Howland's Problem

It was the author's original purpose, before discovering the errors in Pellett's analysis, to extend this work to the case of the circular hole in an infinite strip having finite width and having no loadings on the finite boundaries. The first requirement for a solution to this problem is to determine the membrane stress distribution.

Since Howland (5) has discussed this problem and provided an iterative solution for the membrane stress distribution, it is appropriate to call this problem by his name, Howland's problem. However, his solution was not considered very tractable for use in an energy approach to the buckling problem. Howland's solution involves the sum of a finite number of terms, ϕ_m , each of which is an Airy stress function, to approximate the Airy stress function for the problem. In theory each of the ϕ_m 's contains an infinite number of terms; however, in practice only a finite number of terms can be used. Howland's solution becomes invalid at relatively small distances from the hole (approximately one plate width or less). This difficulty could be eliminated by using a larger number of ϕ_m 's and more terms in each of the ϕ_m 's. However, his method of determining the coefficients is quite involved. For these reasons, several attempts were made at solving Howland's problem by other methods.

Neither of the two approaches to this problem which are presented in this appendix was successful. It is hoped that the information presented here may prove useful to someone else who may be interested in the solution of this problem.

Description of the Problem

Consider a strip of finite width, $2b$, infinitely extended in the other direction, pierced by a hole at the center, and subjected to uniform tensile stress, S , at infinity as shown in figure (10).

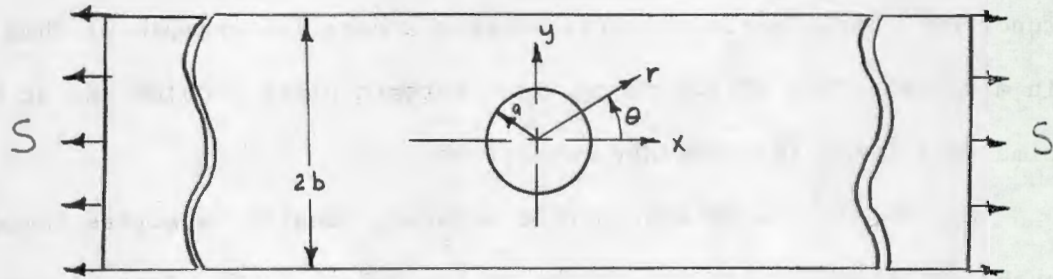


Figure 10

It is convenient to employ two sets of coordinates: x and y are cartesian coordinates and r and θ are cylindrical coordinates. The boundaries in the y direction are at $\pm b$. Without loss of generality all dimensions can be considered in units of hole radius. This reduces all quantities to dimensionless ratios with the hole radius equal to unity.

The boundary conditions for this problem are as follows:

$$(\sigma_r)_{r=1} = 0 \quad (D-1)$$

$$(\tau_{r\theta})_{r=1} = 0 \quad (D-2)$$

$$(\sigma_y)_{y=\pm b} = 0 \quad (D-3)$$

$$(\tau_{xy})_{y=\pm b} = 0 \quad (D-4)$$

$$(\sigma_x)_{x=\infty} = S \quad (D-5)$$

$$(\tau_{xy})_{x=\infty} = 0 \quad (D-6)$$

In each of the attacks on the problem which will be described below, the point of departure is assuming a stress function from which the stress components may be determined, as with an Airy stress function, so as to satisfy the laws of equilibrium. However, instead of requiring this function to satisfy the biharmonic equation, it is selected so as to

minimize the membrane strain energy in the plate, a number of disposable parameters having been introduced so as to permit minimization and so as to permit a stress distribution approaching the correct distribution. The boundary conditions are satisfied by including corresponding terms, multiplied by undetermined factors (Lagrange multipliers) in the strain energy function. Budiansky and Hu (1) present a detailed example of this method in minimizing the strain energy in a certain plate problem and at the same time satisfying the boundary conditions.

In the first approach to this problem, consider a stress function such that the boundary conditions on the hole and the edges of the strip are satisfied. The Lagrangian multiplier is to be used to satisfy the boundary conditions at $x = \infty$. Such a function might be as follows:

$$\phi(r, \theta) = f(r, \theta) \cdot g(r, \theta) \cdot h(r, \theta) \quad (D-7)$$

where $f(r, \theta)$ is such that the boundary conditions at the edge of the plate are satisfied, and $g(r, \theta)$ is such that the boundary conditions on the hole are satisfied. The function $h(r, \theta)$ is to be general enough that the actual stress distribution in the plate can be closely approximated. The functions selected were

$$f = \left(1 - \frac{r^2 \sin^2 \theta}{b^2}\right)^2 = \left(1 - \frac{y^2}{b^2}\right)^2 \quad (D-8)$$

$$g = \left(1 - \frac{1}{r}\right)^2 \quad (D-9)$$

$$h = \sum_{n=0}^N \sum_{m=0}^M A_{nm} r^{-n} \cos 2m\theta \quad (D-10)$$

It can be shown that the boundary conditions D-1, 2, 3, 4, are in fact satisfied by the above choice of the functions f and g . The function h appears to be quite general, considering that positive exponents of r must be excluded in order for the resultant stresses to remain finite as x (and r) approach infinity.

The boundary conditions at $x = \infty$, namely equations D-5 and D-6, are yet to be satisfied. In the limit as x approaches infinity σ_r approaches σ_x and r approaches x . The boundary condition D-5 becomes

$$(\sigma_r)_{r=\infty} = S \quad (D-11)$$

This results in a relatively simple equation in terms of the stress function ϕ .

$$-\frac{b^2}{4} \sum_{m=0}^M A_{om} = S \quad (D-12)$$

The difficulty in the solution arises from the fact that the stress at $x = \infty$ is dependent upon y . The stress varies in a parabolic manner as shown in Figure 11.

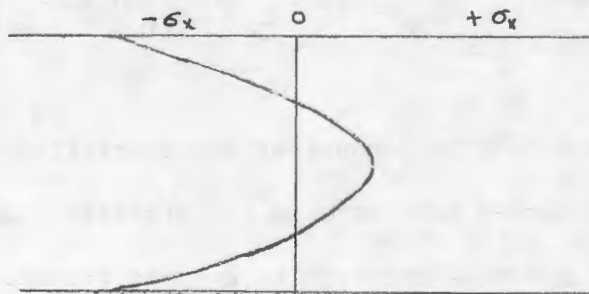


Figure 11

When the stress is integrated from $y = -b$ to $y = +b$ in order to determine the average value of σ_x , the result is zero, i.e., there is no net loading on the strip, just local stresses at $x = \pm \infty$.

It is interesting to note that while the above expression for ϕ appears quite complicated, that it can be reduced to a very compact form. If the functions f and g are expressed as binomial expansions and $\cos 2m\theta$ is expressed as powers of $\sin \theta$, equations D-8, D-9, and D-10 become respective

$$f = \sum_{i=0}^2 \binom{2}{i} \left(\frac{-r^2 \sin^2 \theta}{b^2} \right)^i \quad (D-13)$$

$$g = \sum_{j=0}^2 \binom{2}{j} (-r)^{-j} \quad (D-14)$$

$$h = \sum_{n=0}^N \sum_{m=0}^M \sum_{k=0}^m A_{nm} C_{mk} r^{-n} \sin^{2k} \theta \quad (D-15)$$

where

$$\begin{cases} C_{m0} = 1 \\ C_{mk} = (-1)^k m 2^{2k} (m+k-1)! / (2k)! (m-k)! \end{cases}$$

Let $\sum^{(5)}$ denote $\sum_{i=0}^2 \sum_{j=0}^2 \sum_{n=0}^N \sum_{m=0}^M \sum_{k=0}^m$

and
$$D_{(5)} = A_{nm} C_{mk} \binom{2}{i} \binom{2}{j} (-1)^{i+j}$$

The resulting expression for ϕ is

$$\phi = \sum D_{(5)} r^{2i-j-n} b^{-2i} \sin^{2k+2i} \theta \quad (D-16)$$

The form of equation D-16 considerably simplifies the job of differentiating ϕ in order to determine the stresses. It also simplifies the expression for the strain energy to a great extent. This form is also well suited for use in a computer solution to the problem.

Howland's method of solution involves starting with a stress function ϕ_0 , which results in the Kirsch solution for stress. This would result in σ_y and τ_{xy} stresses on the boundaries at $y = \pm b$. He then adds a stress function ϕ_1 , such that the stress on the edges is reduced to zero but there would have to be loadings on the boundary of the hole. Howland then adds another stress function ϕ_2 , which reduces the stresses at the hole to zero but introduces loadings at the edges of the plate. However, the stresses at the edge due to $\phi_0 + \phi_1 + \phi_2$ are different than those due to ϕ_0 alone. He continues this process until the stresses on the edges in the neighborhood of the hole are reduced to an acceptably low value.

Howland's solution is of the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \dots + \phi_{25} \quad (D-17)$$

where each ϕ_{2k} is of the form

$$\phi_{2k} = -A \ln r + \sum_{n=1}^{\infty} (B_n r^{-2n} + C_n r^{-2n+2}) \cos 2n\theta \quad (D-18)$$

and each ϕ_{2k+1} is of the form

$$\phi_{2k+1} = \sum_{n=0}^{\infty} (D_n r^{2n} + E_n r^{2n+2}) \cos 2n\theta \quad (D-19)$$

Rather than use Howland's method of solving for the coefficients of each of the ϕ 's the following stress function is proposed.

$$\phi = \phi_0 + \phi_1 + \phi_2 \quad (D-20)$$

where ϕ_0 is the same as Howland's ϕ_0 . ϕ_1 and ϕ_2 are of the following form

$$\phi_1 = \sum_{n=0}^{\infty} (D_n r^{2n} + E_n r^{2n+2}) \cos 2n\theta \quad (D-21)$$

$$\phi_2 = \sum_{n=1}^{\infty} (B_n r^{-2n} + C_n r^{-2n+2}) \cos 2n\theta + A \ln r \quad (D-22)$$

This ϕ is equivalent to Howland's ϕ . The intention is to solve for the unknown coefficients using energy methods rather than Howland's alternating procedure. Since ϕ_0 satisfies the boundary condition at x equal to infinity, the boundary conditions of the sum $\phi_1 + \phi_2$ are

$$(\sigma_x)_{x=\infty} = 0 \quad (\tau_{xy})_{x=\infty} = 0$$

In order to satisfy these conditions all of the coefficients D_n and

E_n must be equal to zero, i.e., $\phi_1 = 0$.

Since ϕ_0 satisfies the conditions for zero tractions on the edge of the hole, ϕ_2 must also satisfy these conditions. In order for ϕ_2 to satisfy these conditions the coefficient A and all the coefficients B_n and C_n must be equal to zero, i.e., $\phi_2 = 0$.

Hence, $\phi = \phi_0$, which we know does not satisfy the boundary conditions at $y = \pm b$.

Suggestions for solving Howland's Problem

Since as a practical matter it is necessary to truncate the series used in Howland's solution, his solution provides valid results only for a limited range of values of x . Accordingly, the so-called "finite element method" applied to a finite plate should provide as good a solution to Howland's problem as Howland himself was able to provide.

One is motivated to deal with the problem of the plate of finite width because all plates in actual practice have finite width. Inasmuch as all real plates also have finite length, it can be convincingly argued that the finite element method applied to a finite plate would produce more meaningful results than would a solution to Howland's problem, however the latter might be achieved.

Those familiar with finite element techniques contend that the finite element method can be employed to solve the buckling problem directly. Also, with what would appear to be developments within the state of the art, it would be possible not only to deal with the incipient buckling problem, but also to deal with the post-buckling problem in which the membrane stress distribution is coupled with the lateral deflection.

APPENDIX E

Computer Program Listings

The computer program listings on the following pages were used in the solution of the problem. All programs are written in FORTRAN IV for use on an IBM System 360 computer.

The program was originally written by Pellett (7); however, modifications were made to the program to reflect changes in the solution to the problem as noted previously.

The plate thickness, N , and M are read into the computer by data card (see comment cards regarding input requirements). Young's modulus is established by card 0010 in the main program. Poisson's ratio is established by card 0011 in the main program and card 0003 in subroutine BOUND.

The output is in the form of a column vector EIVU which contains the values of S_{cr} in psi for the parameters of plate thickness, Poisson's ratio, Young's modulus, M , and N . This is followed by a deflection contour map for the lowest value of tensile buckling stress.

60

```

0078      C      TEST TO SEE IF SINE OR COSINE TERMS OR PRESENT AND SUM ACCORDINGLY
      C      IF (JOY-2) 10,10,9
      C
      C      THE A MATRIX REPRESENTS THE BENDING ENERGY INTEGRAL
0079      10 A(JROW,JCOL)=USA*THICK
0080      B(JROW,JCOL)=U3T*PIG10
0081      GO TO 8
0082      9 A(JROW,JCOL)=USB*THICK
0083      B(JROW,JCOL)=UBTB*PIG10
0084      8 CONTINUE
0085      990 FORMAT(1H,9HASYMMETRY=,2X,I3,3X,2HN=,2X,I3,3X,2HM=,2X,I3,3X,6HTER
      1MS=,2X,I3,3X,10HTHICKNESS=,G12.6/)
0086      12 CONTINUE
      C IF A 'GO TO 800' CARD FOLLOWS, MATRIX ELEMENT PRINT IS SUPPRESSED
0087      GO TO 800
0088      887 IF(NEWPAG-50)23,25,25
0089      23 WRITE(6,990) JOY,N,M,INIT,THICK
0090      WRITE(6,910)
0091      910 FORMAT(1H,13H I J K L 5X,4HCLCJ,7X,4HSLSJ,7X,6HSLJJC2,5X,6HCLCJ,
      1JC2,5X,6HSLJJC2,5X,6HSLJJC2,5X,7HSTRETCH,9X,4HEND,7/)
0092      NEWPAG = 0
0093      23 WRITE(6,920) I,J,K,L,CLCJ,SLSJ,SLSJC2,CLCJC2,SLSJC2,SJS2CL,A(JROW,J
      1COL),B(JROW,JCOL)
0094      920 FORMAT(1X,4(I2,1X),2X,6(G10.3,1X),2X,2(G15.8)
0095      NEWPAG=NEWPAG+1
0096      800 CONTINUE
0097      WRITE(6,990) JOY,N,M,INIT,THICK
      C
      C      SOLVE THE EIGENVALUE PROBLEM AX+B=0
0098      CALL JUGGLE (INIT)
0099      DO 225 I=1,INIT
0100      WRITE(6,113) A(I,3)
0101      DO 225 J=1,INIT
0102      225 EIVR(I,J)=EIVU(I,J)
0103      WRITE(6,113) EIVU(I,3)
0104      WRITE(6,990) JOY,N,M,INIT,THICK
0105      BETWEEN=TIME(0)*.01
0106      TIME=BETWEEN-START
0107      WRITE(6,995) TIME
0108      NULL=0
0109      DO 220 I=1,INIT
      C
      C      DETERMINE WHICH EIGENVALUE S ARE POSITIVE
      C      STORE THE POSITIVE EIGEN VALUES IN TEMP A
0110      IF (EIVU(I))220,220,221
0111      221 NULL=NULL+1
0112      TEMP A=NULL+EIVU(I)
0113      DO 222 J=1,INIT
0114      222 EIVR(J,NULL)=EIVR(J,I)
0115      220 CONTINUE
      C
      C      FIND THE SMALLEST POSITIVE EIGENVALUE
0116      BAT=1.D+70
0117      DO 110 I=1,NULL
0118      IF(TEMP A(I)-BAT) 100,100,110
0119      100 BAT=TEMP A(I)
0120      DO 109 J=1,INIT
0121      EIVU(J)=EIVR(J,I)
0122      109 CONTINUE
0123      110 CONTINUE
0124      TEMP A(I)=BAT
0125      DO 111 J=1,INIT
0126      EIVR(J,I)=EIVU(J)
0127      WRITE(6,113) EIVU(J)
0128      111 CONTINUE
0129      WRITE(6,112) TEMP A(I)
0130      112 FORMAT(1H,MIN CRITICAL STRESS=,G12.5,'PSI')
0131      WRITE(6,113)(EIVR(J,I),J=1,INIT)
0132      113 FORMAT(20,10)
0133      IF (NULL.EQ.0) GO TO 844
0134      NULL=1
0135      DO 223 NO=1,NULL
      C
      C      PRINT THE MODE SHAPES FOR THE POSITIVE EIGENVALUES.
0136      CALL PICTUR(ND,25)
0137      223 CONTINUE
0138      BETWEEN=TIME(0)*.01
0139      TIME=BETWEEN-START
0140      WRITE(6,995) TIME
0141      844 CONTINUE
0142      995 FORMAT(1X,'ELAPSED TIME = ',G13.5,'SECONDS')
0143      STOP
0144      END

```

```

0001 SUBROUTINE BOUND(N,M,X,Y)
0002 C THIS SUBROUTINE FINDS THE VALUES OF X AND Y FOR A GIVEN N AND M
0003 C SO THAT THE BOUNDARY CONDITIONS WILL BE SATISFIED.
0004 VNU=0.300
0005 OMEN1=N*(N+1)-N*VNU-M**2*VNU
0006 OMEN3=(N+1)*(N+2)-(N+1)*VNU-M**2*VNU
0007 OMEN2=(N+2)*(N+3)-(N+2)*VNU-M**2*VNU
0008 SHE1=-N*(N+1)*(N+2)+N*(N+1)+N*(2-VNU)*M**2*N+(3-VNU)*M**2
0009 SHE2=-(N+1)*(N+2)*(N+3)+(N+1)*(N+2)+(N+1)*(2-VNU)*M**2*(N+1)+(3-VN
0010 1U)*M**2
0011 SHE3=-(N+2)*(N+3)*(N+4)+(N+2)*(N+3)+(N+2)*(2-VNU)*M**2*(N+2)+(3-VN
0012 1U)*M**2
0013 X=(-OMEN1*SHE3+OMEN3*SHE1)/(OMEN2*SHE3-OMEN3*SHE2)
0014 Y=(-OMEN2*SHE1+OMEN1*SHE2)/(OMEN2*SHE3-OMEN3*SHE2)
0015 RETURN
0016 END

```

```

0001 SUBROUTINE CCGALV(N,X,Y,Z)
0002 C THIS SUBROUTINE FINDS THE VALUE OF THE INTEGRAL OF
0003 C COSINE (IXX*THETA) COSINE (IYY*THETA) COSINE (IZZ*THETA)
0004 C AS THETA RANGES FROM ZERO TO TWO PI
0005 C IMPLICIT REAL*8 (A-H,O-Z)
0006 B1=0.000
0007 B2=0.000
0008 B3=0.000
0009 B4=0.000
0010 IF (IXX-IYY-IZZ)1,2,1
0011 2 B1=.500
0012 1 IF (IXX+IYY+IZZ)3,4,3
0013 4 B2=.500
0014 3 IF (IXX-IYY+IZZ)5,6,5
0015 6 B3=.500
0016 5 IF (IXX+IYY-IZZ)7,8,7
0017 8 B4=.500
0018 7 VALUE=(B1+B2+B3+B4)*3.141592653589793238
0019 RETURN
0020 END

```

```

0001 SUBROUTINE SSC (VALUE,IXX,IYY,IZZ)
0002 C THIS SUBROUTINE FINDS THE VALUE OF THE INTEGRAL
0003 C SINE (IXX*THETA) SINE (IYY*THETA) COSINE (IZZ*THETA)
0004 C AS THETA RANGES FROM ZERO TO TWO PI
0005 C IMPLICIT REAL*8 (A-H,O-Z)
0006 B1=0.000
0007 B2=0.000
0008 B3=0.000
0009 B4=0.000
0010 VALUE=0.000
0011 IF (IXX-IYY)10,9,10
0012 10 IF (IXX-IYY-IZZ)1,2,1
0013 2 B1=.500
0014 1 IF (IXX+IYY+IZZ)3,4,3
0015 4 B2=.500
0016 3 IF (IXX-IYY+IZZ)5,6,5
0017 6 B3=.500
0018 5 IF (IXX+IYY-IZZ)7,8,7
0019 8 B4=.500
0020 7 VALUE=(B1+B2+B3+B4)*3.141592653589793238
0021 RETURN
0022 END

```



```

0001      C      SUBROUTINE PICTURE (NO,IRAO)
0002      THIS SUBROUTINE FORMS THE DEFLECTION MATRIX OF REPRESENTATIVE
0003      POINTS ON THE PLATE FOR USE IN SUBROUTINE CONTR
0004      IMPLICIT REAL*8 (A-H,O-Z)
0005      REAL*4 DATA,DUMM
0006      COMMON/BANE/ X(100),Y(100)
0007      COMMON/TAME/START
0008      COMMON/CHUNK/ DATA(160,110),DUMM(8000),A(80,80)
0009      COMMON/MUCK/EIVU(100),TEMPA(100)
0010      COMMON/PASS/MCOUNT(100),JOY,N,M,INIT,THICK
0011      RADIUS=IRAO
0012      WRITE (6,990) JOY,N,M,INIT,THICK
0013      WRITE(6,90) TEMPA(NO)
0014      DO 125 MID=1,N
0015      DO 125 NID=1,M
0016      JROW=(NID-1)*M+MID
0017      EIVU(JROW)=A(JROW,NO)
0018      WRITE(6,91)NID,MCOUNT(MID),A(JROW,NO)
0019      125 CONTINUE
0020      DO 210 IY=1,110
0021      WY=IY
0022      WY=WY-55,DO
0023      DO 210 IX=1,160
0024      EX=(IX-80)/.800
0025      OVARE=RADIUS/(DSQRT(EX*EX+WY*WY))
0026      WWW=0.00
0027      IF(1.00-OVARE)210,126,126
0028      126 CONTINUE
0029      THETA=ATAN2(WY,EX)
0030      PARE=1.00
0031      DO 213 NID=1,N
0032      PARE=PARE*OVARE
0033      N1=NID+1
0034      N2=NID+2
0035      N3=NID+3
0036      DO 213 MID=1,M
0037      JROW=(NID-1)*M+MID
0038      IF (JOY-2)214,214,215
0039      214 WWW=WWW+PARE*(1.00+OVARE*(X(JROW)+OVARE*Y(JROW)))* OCOS(MCOUNT(MI
0040      ID)*THETA)*A(JROW,NO)
0041      GO TO 213
0042      215 WWW=WWW+PARE*(1.00+OVARE*(X(JROW)+OVARE*Y(JROW)))* DSIN(MCOUNT(MI
0043      ID)*THETA)*A(JROW,NO)
0044      213 CONTINUE
0045      DATA(IX,IY)=WWW
0046      210 WRITE (6,990) JOY,N,M,INIT,THICK
0047      WRITE(6,92)
0048      92 FORMAT(1X,' MAP OF DEFLECTION')
0049      CALL CONMAP
0050      90 FORMAT(1X,' CRITICAL BUCKLING STRESS = ',G12.5,' PSI'//1X,' COEFFICI
0051      ENTS OF THE SERIES'//)
0052      91 FORMAT(1X,' N= ',I2,' M= ',I2,' A(N,M) = ',G16.6)
0053      990 FORMAT(1H1,9HSYMMETRY=,2X,I3,3X,2HN=,2X,I3,3X,2HM=,2X,I3,3X,6HTER
0054      MS=,2X,I3,3X,10HTHICKNESS=,G12.6//)
0055      RETURN
0056      444 FORMAT(1X,' ELAPSED TIME = ',G13.5,' SECONDS')
0057      END

```

```

0001      C      SUBROUTINE CONMAP
0002      THIS SUBROUTINE WILL PRODUCE A MAP OF THE NUMBERS CONTAINED
0003      IN THE ARRAY A
0004      160 IS THE NUMBER OF LINES OF THE MAP
0005      110 IS THE WIDTH OF A LINE
0006      COMMON/CHUNK/ A(160,110),DUMM(8000),DUM2(12800)
0007      DIMENSION IS(23),LINE(110)
0008      DATA IS/1H1,1H0,1H ,1H1,1H ,1H2,1H ,1H3,1H ,1H4,1H ,1H5,1H ,1H6,
0009      1H ,1H7,1H ,1H8,1H ,1H9,1H ,1H0,1HM)
0010      FIND THE MAX AND MIN DEFLECTIONS
0011      CC=-1.E70
0012      BB= 1.E70
0013      DO 5 I=1,160
0014      DO 5 J=1,110
0015      CC=AMAX1(CC,A(I,J))
0016      BB=AMIN1(BB,A(I,J))
0017      5 CONTINUE
0018      WRITE(6,6) CC,BB
0019      6 FORMAT(17H MAX DEFLECTION =,E15.5,17H MIN DEFLECTION =,E15.5)
0020      WRITE THE DEFLECTION
0021      DO 7 I=1,160
0022      DO 4 J=1,110
0023      IF(A(I,J)-BB) 30,30,31
0024      30 JK=1
0025      GO TO 34
0026      31 IF(CC-A(I,J)) 32,32,33
0027      32 JK=23
0028      GO TO 34
0029      33 JK=20*(A(I,J)-BB)/(CC-BB)+2.5
0030      34 CONTINUE
0031      LINE(J)=IS(JK)
0032      7 WRITE(6,8) LINE
0033      8 FORMAT(4X,110A1)
0034      RETURN
0035      END

```

```

0001      SUBROUTINE PRINN (N,M,NPAGE)
C      THIS SUBROUTINE PRINTS THE MATRIX A,B,OR EIVR, DEPENDING ON
C      WHETHER N IN THE CALL ARGUMENT IS 1,2, OR 3. IF NPAGE EQUALS ZERO
C      NO PAGE HEADINGS WILL BE PRODUCED. IF NPAGE EQUALS 1 A NEW PAGE
C      WILL BE STARTED. THIS SUBROUTINE FINDS UTILITY IN DEBUGGING
C      WHEN MATRIX PRINTOUT IS NEEDED.
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/NUCK/EIVU(100),TEMPA(100)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      IF (NPAGE.EQ.1) WRITE (6,99)
0002      99 FORMAT ('1')
0003      GO TO (1,2,3,14,15),N
0004      1 WRITE(6,91)
0005      91 FORMAT('1','MATRIX IN A'//)
0006      GO TO 4
0007      2 WRITE (6,92)
0008      92 FORMAT('1','MATRIX IN B'//)
0009      GO TO 4
0010      3 WRITE (6,93)
0011      93 FORMAT('1','MATRIX IN EIVR'//)
0012      GO TO 4
0013      14 WRITE(6,94)
0014      94 FORMAT('1','MATRIX IN EIVU'//)
0015      18 WRITE (6,88)((J,EIVU(J)),J=1,M)
0016      GO TO 100
0017      15 WRITE (6,85)
0018      85 FORMAT('1','MATRIX IN TEMPA'//)
0019      19 WRITE (6,88)((J,TEMPA(J)),J=1,M)
0020      GO TO 100
0021      4 DO 8 I=1,M
0022      GO TO (5,6,7),N
0023      5 WRITE (6,94)(A(I,J),J=1,M)
0024      WRITE(6,95)
0025      GO TO 8
0026      6 WRITE (6,94)(B(I,J),J=1,M)
0027      WRITE(6,95)
0028      GO TO 8
0029      7 WRITE (6,94)(EIVR(I,J),J=1,M)
0030      WRITE(6,95)
0031      GO TO 8
0032      8 CONTINUE
0033      100 CONTINUE
0034      RETURN
0035      88 FORMAT(1X,'EIG('1,2,')= ',G15.7)
0036      94 FORMAT (1X,10G12.5)
0037      95 FORMAT(1X)
0038      END
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0001      SUBROUTINE JUGGLE (LINK)
C      THIS SUBROUTINE CONVERTS A MATRIX EIGENVALUE OF THE FORM (AX+B)Y=0
C      TO THE FORM (IX+C)Z=0 WHERE A,B,C ARE COEFFICIENT MATRICES
C      X IS THE EIGENVALUE MATRIX, AND Y & Z ARE THE EIGENVECTOR MATRICES
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/NUCK/EIVU(100),TEMPA(100)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      DO 1 I=1,LINK
      DO 1 J=1,LINK
      EIVR(I,J)=A(I,J)
      A(I,J)=B(I,J)
      1 B(I,J)=EIVR(I,J)
C      CALCULATE AND PRINT THE TRACE OF A
      TR=0.00
      DO 11 I=1,LINK
      11 TR=TR+A(I,I)
      WRITE(6,12) TR
      12 FORMAT(8H,TRACE =,G15.7)
      CALL JACVAT (LINK)
      CALL PRINN(4,LINK,0)
C      SUM THE EIVU'S
      SUM=EIVU(1)
      DO 20 I=2,LINK
      20 SUM =SUM+EIVU(I)
      WRITE(6,21) SUM
      21 FORMAT(11H,SUM EIVU =,G15.7)
      DO 2 J=1,LINK
      DO 2 I=1,LINK
      SUM=0.00
      DO 3 JOKE=1,LINK
      3 SUM=EIVR(JOKE,I)*B(JOKE,J)+SUM
      2 A(I,J)=SUM
      DO 4 J=1,LINK
      DO 4 I=1,LINK
      SUM=0.00
      DO 5 JOKE=1,LINK
      5 SUM=A(I,JOKE)*EIVR(JOKE,J)+SUM
      4 B(I,J)=SUM
      DO 6 I=1,LINK
C      CHECK FOR NEGATIVE EIGENVALUES. IF ANY ARE NEGATIVE RETURN
C      TO THE CALLING PROGRAM
      IF(EIVU(1)) 30,30,31
      30 WRITE (6,32)
      32 FORMAT(T20,'THE VALUE OF AN EIGENVALUE IS LESS THAN ZERO')
      RETURN
      31 CONTINUE
      TEMPA(1)=DSORT(EIVU(1))
      6 EIVU(1)=1.00/DSORT(EIVU(1))
      DO 7 I=1,LINK
      DO 7 J=1,LINK
      A(I,J)=B(I,J)*EIVU(1)*EIVU(J)
      7 B(I,J)=EIVR(I,J)
      CALL JACVAT(LINK)
      DO 8 I=1,LINK
      DO 8 J=1,LINK
      SUM=0.00
      DO 9 JOKE=1,LINK
      9 SUM=B(JOKE,I)*EIVR(JOKE,J)+SUM
      8 A(I,J)=SUM*TEMPA(1)
      DO 10 I=1,LINK
      10 EIVU(I)=1.00/EIVU(1)
      CALL PRINN(4,LINK,0)
      RETURN
      END
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0001 SUBROUTINE JACVAT (N)
      THIS SUBROUTINE SOLVES THE MATRIX EIGENVALUE PROBLEM  $(X-A)Y=0$ ,
      WHERE X IS THE EIGENVALUE MATRIX IN EIVU, Y IS THE EIGENVECTOR MATRIX
      IN EIVR, AND A IS THE COEFFICIENT MATRIX IN A
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      NOYES=1
      IF(N-1)20,23,21
      20 WRITE (6,12) N
      22 FORMAT(4H N=,I3,6H IS TOO SMALL. LIMIT IS 1. RETURN TO CALLING
      1 ROUTINE FROM JACVAT. )
      RETURN
      23 WRITE (6,24) A(1,1)
      24 FORMAT(24H IN JACVAT, MATRIX A =,D14.6)
      RETURN
      21 IF(N-160)1,1,3
      3 WRITE(6,2) N
      2 FORMAT(6,2) N
      2 IF(N-15,70H IS TOO LARGE. LIMIT IS 160. RETURN TO CALLING
      1 ROUTINE FROM JACVAT. )
      RETURN
      1 IF(NOYES)99,102,99

      SET THE MATRIX EIVR EQUAL TO THE IDENTITY MATRIX

      99 CONTINUE
      DO 100 J=1,N
      DO 100 I=1,N
      100 EIVR(I,J)=0.000
      101 EIVR(J,J)=1.000

      CHECK SYMMETRY OF THE GIVEN MATRIX A. IF TWO SYMMETRIC ELEMENTS ARE
      NOT EQUAL, REPLACE EACH OF THE UNEQUAL ELEMENTS WITH THE MEAN OF THE
      TWO ELEMENTS, PRINT OUT A MESSAGE, AND CONTINUE WITH THE NEW MATRIX.

      102 ATOP=0.000
      DO 103 J=1,N
      DO 103 I=1,J
      IF(A(I,J)-A(J,I))90,103,90
      90 WRITE(6,106) N,N
      106 FORMAT(14H IN JACVAT (A(I,1H,,I3,3H)),)
      WRITE (6,108) I,J,J,I
      108 FORMAT(3H A(I,1H,,I3,10H) AND A(I,1H,,I3,54H) WERE UNEQUAL,
      150 THEY WERE REPLACED WITH THEIR MEAN )
      90 CONTINUE
      A(I,J)=.5*(A(I,J)+A(J,I))
      A(J,I)=A(I,J)
      103 CONTINUE

      FIND THE ABSOLUTELY LARGEST ELEMENT OF A

      IF(ATOP-DABS(A(I,J)))104,111,111
      104 ATOP=DABS(A(I,J))
      111 CONTINUE
      112 EIVU(J)=A(I,J)
      IF(ATOP)109,109,113
      109 WRITE (6,110)
      110 FORMAT(26H IN JACVAT, MATRIX A = 0 )
      RETURN

      CALCULATE THE STOPPING CRITERION -- DSTOP

      113 AVGF=DFLOAT(N*(N-1))*55
      D=0.000
      DO 114 JJ=2,N
      DO 114 II=2,JJ
      S=A(II-1,JJ)/ATOP
      114 D=S*D
      DSTOP=(1.D-12)*D

      CALCULATE THE THRESHOLD, THRSH

      THRSH = DSORT(D/AVGF)*ATOP

      START A SWEEP

      115 IFLAG=0
      DO 130 JCOL=2,N
      JCOL1=JCOL-1
      DO 130 IROW=1,JCOL1
      AIJ=A(IROW,JCOL)

      COMPARE THE OFF-DIAGONAL ELEMENT WITH THRSH

      IF(DABS(AIJ)-THRSH)130,130,117
      117 AIJ=A(IROW,IROW)
      AJJ=A(JCOL,JCOL)
      S=AJJ-AIJ

      CHECK TO SEE IF THE CHOSEN ROTATION IS LESS THAN THE ROUNDING ERROR.
      IF SO, THEN DO NOT ROTATE.

      IF(DABS(AIJ)-1.D-17*DABS(S))130,130,118
      118 IFLAG=1

      IF THE ROTATION IS VERY CLOSE TO 45 DEGREES, SET SIN AND COS
      TO 1/(ROOT 2).

      IF(1.D-10*DABS(AIJ)-DABS(S))116,119,119
      119 S=DSORT(.500)
      C=S
      GO TO 120

      CALCULATION OF SIN AND COS FOR ROTATION THAT IS NOT VERY CLOSE
      TO 45 DEGREES

      116 T=AIJ/S
      S=0.25/DSORT(0.25+T*T)
      COS = C, SIN= S
      C=DSORT(0.5+S)
      S=2.*T*S/C

      CALCULATION OF THE NEW ELEMENTS OF MATRIX A

      120 DO 121 I=1,IROW
      T=A(I,IROW)
      U=A(I,JCOL)
      A(I,IROW)=C*T-S*U
      121 A(I,JCOL)=S*T+C*U
      I2=IROW+2
      IF(I2-JCOL)127,127,123
      127 CONTINUE
      DO 122 I=I2,JCOL
      T=A(I-1,JCOL)
      U=A(IROW,I-1)
      A(I-1,JCOL)=S*U+C*T
      122 A(IROW,I-1)=C*U-S*T
      123 A(JCOL,JCOL)=S*AIJ+C*AJJ
      A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)
      DO 124 J=JCOL,N
      T=A(IROW,J)
      U=A(JCOL,J)
      A(IROW,J)=C*T-S*U
      124 A(JCOL,J)=S*T+C*U

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CCCC ROTATION COMPLETED
      SEE IF EIGENVECTORS ARE WANTED BY USER
0089 IF(INOYES)131,126,131
0090 131 CONTINUE
0091 DO 128 I=1,N
0092 T=EIVR(I,IC1)*N
0093 EIVR(I,IC1)=C*T+EIVR(I,JCOL)*S
0094 125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C

C
      CALCULATE THE NEW NORM D AND COMPARE WITH DSTOP
0095 126 CONTINUE
0096 S=A(I,J)/ATOP
0097 D=D-S
0098 IF(D-DSTOP)1260,129,129

C
      RECALCULATE DSTOP AND THRSH TO DISCARD ROUNDING ERRORS
0099 1260 D=0.000
0100 DO 128 JJ=2,N
0101 DO 128 II=2,JJ
0102 D=A(I,JJ)/ATOP
0103 128 D=S+D
0104 DSTOP=(1,D-12)*D
0105 THRSH=D*ORT(D/AVGF)*ATOP
0106 130 CONTINUE
0107 IF(FLAG)115,134,115

C
      ENDING ROUTINE -- RESET MATRIX A AND PLACE EIGENVALUES IN EIVU
0108 134 T=A(1,1)
0109 A(1,1)=EIVU(1)
0110 EIVU(1)=T
0111 DO 132 J=2,N
0112 T=A(1,J)
0113 A(1,J)=EIVU(J)
0114 EIVU(J)=T
0115 DO 132 I=2,J
0116 A(I-1,J)=A(J,I-1)
0117 132 RETURN
0118 END

0001 C
      SUBROUTINE TERMS(IAD,K)
      THIS SUBROUTINE FINDS THE INTEGRATED VALUE OF THE TERMS OF THE
      STRETCH AND BENDING INTEGRALS FOR PLATE ENERGY
      IAD=1,11 REAL*(A-H,D-Z)
      COMMON/PAST/JROW,JCOL
      COMMON/BANE/XX(100),YY(100)
      COMMON/NUMBER/US1,US2,US3,US4,US5,US6,US7A,US7B,US8A,US8B,US9A,
      US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
      2 US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
      2 US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
      DIMENSION SUB(9)
      X=XX(JCOL)
      XSTAR=XX(JROW)
      Y=YY(JCOL)
      YSTAR=YY(JROW)
      US1=0.00
      US2=0.00
      US3=0.00
      US4=0.00
      US5=0.00
      US6=0.00
      US7A=0.00
      US7B=0.00
      US8A=0.00
      US8B=0.00
      US9A=0.00
      US9B=0.00
      UB1=0.00
      UB2A=0.00
      UB2B=0.00
      UB3A=0.00
      UB3B=0.00
      UB4=0.00
      UB5A=0.00
      UB5B=0.00
      UB6=0.00
      UB7=0.00
      UB8A=0.00
      UB8B=0.00
      UB9=0.00
      TOR1=I+K
      TOR2=I+K+1
      TOR3=I+K+2
      TOR4=I+K+3
      TOR5=I+K+4
      TOR6=I+K+5
      TOR7=I+K+6
      TOR8=I+K+7
      DENOM=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM1=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM2=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM3=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM4=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM5=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM6=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM7=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM8=TOR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      USD=TOR*DENOM
      UB0=TOR*DENOM
      SUB(1)=I*(K+1)*(K+1)*DENOM2
      SUB(2)=I*(K+1)*(K+1)*(K+2)*X*YSTAR*DENOM4
      SUB(3)=I*(K+1)*(K+2)*(K+2)*Y*YSTAR*DENOM6
      SUB(4)=I*(K+1)*(K+1)*(K+2)*X*YSTAR*DENOM3
      SUB(5)=I*(K+1)*(K+1)*(K+2)*X*YSTAR*DENOM3
      SUB(6)=I*(K+1)*(K+2)*(K+2)*Y*YSTAR*DENOM4
      SUB(7)=I*(K+1)*(K+2)*(K+2)*X*YSTAR*DENOM4
      SUB(8)=I*(K+1)*(K+2)*(K+2)*Y*YSTAR*DENOM4
      SUB(9)=I*(K+1)*(K+2)*(K+2)*X*YSTAR*DENOM5
      UB1=0
      DO1 KK=1,9
      1 UB1=BUB(KK)+UB1
      BUB(1)=-I*(K+1)*(K+1)*DENOM2
      BUB(2)=-I*(K+1)*(K+2)*X*YSTAR*DENOM3

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0070 SUB(1)=-1*(K+1)*K+3*(Y*DENOM4
0071 SUB(2)=-1*(K+1)*K+2*(Y*STAR*DENOM4
0072 SUB(3)=-1*(K+1)*K+2*(Y*STAR*DENOM4
0073 SUB(4)=-1*(K+1)*K+2*(Y*STAR*DENOM5
0074 SUB(5)=-1*(K+1)*K+2*(Y*STAR*DENOM4
0075 SUB(6)=-1*(K+1)*K+2*(Y*STAR*DENOM5
0076 SUB(7)=-1*(K+1)*K+2*(Y*STAR*DENOM6
0077 DO 2 KK=1,9
0078 2 UB2A=UB2A+SUB(KK)
0079 SUB(1)=-K*(I+1)*DENOM2
0080 SUB(2)=-K*(I+1)*L**2*STAR*DENOM3
0081 SUB(3)=-K*(I+1)*L**2*STAR*DENOM4
0082 SUB(4)=-K*(I+1)*L**2*STAR*DENOM3
0083 SUB(5)=-K*(I+1)*L**2*STAR*DENOM4
0084 SUB(6)=-K*(I+1)*L**2*STAR*DENOM5
0085 SUB(7)=-K*(I+1)*L**2*STAR*DENOM4
0086 SUB(8)=-K*(I+1)*L**2*STAR*DENOM5
0087 SUB(9)=-K*(I+1)*L**2*STAR*DENOM6
0088 DO 3 KK=1,9
0089 3 UB2B=UB2B+SUB(KK)
0090 SUB(1)=-K*(I+1)*L**2*DENOM2
0091 SUB(2)=-K*(I+1)*L**2*STAR*DENOM3
0092 SUB(3)=-K*(I+1)*L**2*STAR*DENOM4
0093 SUB(4)=-K*(I+1)*L**2*STAR*DENOM3
0094 SUB(5)=-K*(I+1)*L**2*STAR*DENOM4
0095 SUB(6)=-K*(I+1)*L**2*STAR*DENOM5
0096 SUB(7)=-K*(I+1)*L**2*STAR*DENOM4
0097 SUB(8)=-K*(I+1)*L**2*STAR*DENOM5
0098 SUB(9)=-K*(I+1)*L**2*STAR*DENOM6
0099 DO 4 KK=1,9
0100 4 UB3A=UB3A+SUB(KK)
0101 SUB(1)=-K*(K+1)*J**2*DENOM2
0102 SUB(2)=-K*(K+1)*J**2*STAR*DENOM3
0103 SUB(3)=-K*(K+1)*J**2*STAR*DENOM4
0104 SUB(4)=-K*(K+1)*J**2*STAR*DENOM3
0105 SUB(5)=-K*(K+1)*J**2*STAR*DENOM4
0106 SUB(6)=-K*(K+1)*J**2*STAR*DENOM5
0107 SUB(7)=-K*(K+1)*J**2*STAR*DENOM4
0108 SUB(8)=-K*(K+1)*J**2*STAR*DENOM5
0109 SUB(9)=-K*(K+1)*J**2*STAR*DENOM6
0110 DO 5 KK=1,9
0111 5 UB3B=UB3B+SUB(KK)
0112 SUB(1)=-K*(K+1)*J**2*STAR*DENOM3
0113 SUB(2)=-K*(K+1)*J**2*STAR*DENOM4
0114 SUB(3)=-K*(K+1)*J**2*STAR*DENOM3
0115 SUB(4)=-K*(K+1)*J**2*STAR*DENOM4
0116 SUB(5)=-K*(K+1)*J**2*STAR*DENOM5
0117 SUB(6)=-K*(K+1)*J**2*STAR*DENOM4
0118 SUB(7)=-K*(K+1)*J**2*STAR*DENOM5
0119 SUB(8)=-K*(K+1)*J**2*STAR*DENOM6
0120 SUB(9)=-K*(K+1)*J**2*STAR*DENOM6
0121 DO 6 KK=1,9
0122 6 UB4=UB4+SUB(KK)
0123 SUB(1)=-J**2*(K+1)*X*DENOM3
0124 SUB(2)=-J**2*(K+1)*X*DENOM3
0125 SUB(3)=-J**2*(K+1)*X*DENOM4
0126 SUB(4)=-J**2*(K+1)*X*DENOM3
0127 SUB(5)=-J**2*(K+1)*X*DENOM4
0128 SUB(6)=-J**2*(K+1)*X*DENOM5
0129 SUB(7)=-J**2*(K+1)*X*DENOM4
0130 SUB(8)=-J**2*(K+1)*X*DENOM5
0131 SUB(9)=-J**2*(K+1)*X*DENOM6
0132 DO 7 KK=1,9
0133 7 UB5A=UB5A+SUB(KK)
0134 SUB(1)=-L**2*(I+1)*STAR*DENOM3
0135 SUB(2)=-L**2*(I+1)*STAR*DENOM4
0136 SUB(3)=-L**2*(I+1)*STAR*DENOM3
0137 SUB(4)=-L**2*(I+1)*STAR*DENOM4
0138 SUB(5)=-L**2*(I+1)*STAR*DENOM5
0139 SUB(6)=-L**2*(I+1)*STAR*DENOM4
0140 SUB(7)=-L**2*(I+1)*STAR*DENOM5
0141 SUB(8)=-L**2*(I+1)*STAR*DENOM6
0142 SUB(9)=-L**2*(I+1)*STAR*DENOM6
0143 DO 8 KK=1,9
0144 8 UB5B=UB5B+SUB(KK)
0145 SUB(1)=-J**2*(K+1)*X*DENOM3
0146 SUB(2)=-J**2*(K+1)*X*DENOM3
0147 SUB(3)=-J**2*(K+1)*X*DENOM4
0148 SUB(4)=-J**2*(K+1)*X*DENOM3
0149 SUB(5)=-J**2*(K+1)*X*DENOM4
0150 SUB(6)=-J**2*(K+1)*X*DENOM5
0151 SUB(7)=-J**2*(K+1)*X*DENOM4
0152 SUB(8)=-J**2*(K+1)*X*DENOM5
0153 SUB(9)=-J**2*(K+1)*X*DENOM6
0154 DO 9 KK=1,9
0155 9 UB6=UB6+SUB(KK)
0156 SUB(1)=-J*(K+1)*L*STAR*DENOM3
0157 SUB(2)=-J*(K+1)*L*STAR*DENOM4
0158 SUB(3)=-J*(K+1)*L*STAR*DENOM3
0159 SUB(4)=-J*(K+1)*L*STAR*DENOM4
0160 SUB(5)=-J*(K+1)*L*STAR*DENOM5
0161 SUB(6)=-J*(K+1)*L*STAR*DENOM4
0162 SUB(7)=-J*(K+1)*L*STAR*DENOM5
0163 SUB(8)=-J*(K+1)*L*STAR*DENOM6
0164 SUB(9)=-J*(K+1)*L*STAR*DENOM6
0165 DO 10 KK=1,9
0166 10 UB7=UB7+SUB(KK)
0167 SUB(1)=-J*(K+1)*X*DENOM3
0168 SUB(2)=-J*(K+1)*X*DENOM3
0169 SUB(3)=-J*(K+1)*X*DENOM4
0170 SUB(4)=-J*(K+1)*X*DENOM3
0171 SUB(5)=-J*(K+1)*X*DENOM4
0172 SUB(6)=-J*(K+1)*X*DENOM5
0173 SUB(7)=-J*(K+1)*X*DENOM4
0174 SUB(8)=-J*(K+1)*X*DENOM5
0175 SUB(9)=-J*(K+1)*X*DENOM6
0176 DO 11 KK=1,9
0177 11 UB8A=UB8A+SUB(KK)
0178 SUB(1)=-L*(I+1)*STAR*DENOM3
0179 SUB(2)=-L*(I+1)*STAR*DENOM4
0180 SUB(3)=-L*(I+1)*STAR*DENOM3
0181 SUB(4)=-L*(I+1)*STAR*DENOM4
0182 SUB(5)=-L*(I+1)*STAR*DENOM5
0183 SUB(6)=-L*(I+1)*STAR*DENOM4
0184 SUB(7)=-L*(I+1)*STAR*DENOM5
0185 SUB(8)=-L*(I+1)*STAR*DENOM6
0186 SUB(9)=-L*(I+1)*STAR*DENOM6
0187 DO 12 KK=1,9
0188 12 UB8B=UB8B+SUB(KK)
0189 SUB(1)=-J*(K+1)*X*DENOM3
0190 SUB(2)=-J*(K+1)*X*DENOM4
0191 SUB(3)=-J*(K+1)*X*DENOM3
0192 SUB(4)=-J*(K+1)*X*DENOM4
0193 SUB(5)=-J*(K+1)*X*DENOM5
0194 SUB(6)=-J*(K+1)*X*DENOM4
0195 SUB(7)=-J*(K+1)*X*DENOM5
0196 SUB(8)=-J*(K+1)*X*DENOM6
0197 SUB(9)=-J*(K+1)*X*DENOM6
0198 DO 13 KK=1,9
0199 13 UB9=UB9+SUB(KK)
0200 SUB(1)=-K*(K+1)*X*DENOM1
0201 SUB(2)=-K*(K+1)*X*DENOM2
0202 SUB(3)=-K*(K+1)*X*DENOM1
0203 SUB(4)=-K*(K+1)*X*DENOM2
0204 SUB(5)=-K*(K+1)*X*DENOM3
0205 SUB(6)=-K*(K+1)*X*DENOM2
0206 SUB(7)=-K*(K+1)*X*DENOM3
0207 SUB(8)=-K*(K+1)*X*DENOM3
0208 SUB(9)=-K*(K+1)*X*DENOM4
0209 DO 14 KK=1,9
0210 14 US1=US1+SUB(KK)
0211 SUB(1)=-K*(K+1)*X*DENOM2
0212 SUB(2)=-K*(K+1)*X*DENOM3
0213 SUB(3)=-K*(K+1)*X*DENOM4
0214 SUB(4)=-K*(K+1)*X*DENOM3
0215 SUB(5)=-K*(K+1)*X*DENOM4

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0219 BUB(9)=((I+1)*(K+2)*Y*STAR*DENOM5
0220 BUB(8)=((I+2)*(K+1)*Y*STAR*DENOM5
0221 BUB(9)=((I+2)*(K+2)*Y*STAR*DENOM6
0222 DO 19 KK=1,9
0223 15 US2=US2+BUB(KK)
0224 BUB(1)=I*(K+1)*X*DENOM4
0225 BUB(2)=I*(K+1)*X*DENOM5
0226 BUB(3)=I*(K+2)*Y*DENOM6
0227 BUB(4)=I*(I+1)*K*X*STAR*DENOM5
0228 BUB(5)=I*(I+1)*(K+2)*Y*X*STAR*DENOM6
0229 BUB(6)=I*(I+2)*K*Y*STAR*DENOM7
0230 BUB(7)=I*(I+2)*K*Y*STAR*DENOM6
0231 BUB(8)=I*(I+2)*(K+1)*Y*STAR*DENOM7
0232 BUB(9)=I*(I+2)*(K+2)*Y*STAR*DENOM8
0233 DO 16 KK=1,9
0234 16 US3=US3+BUB(KK)
0235 BUB(1)=J*L*DENOM
0236 BUB(2)=J*L*X*STAR*DENOM1
0237 BUB(3)=J*L*Y*STAR*DENOM2
0238 BUB(4)=J*L*X*DENOM1
0239 BUB(5)=J*L*X*X*STAR*DENOM2
0240 BUB(6)=J*L*X*Y*STAR*DENOM3
0241 BUB(7)=J*L*Y*DENOM2
0242 BUB(8)=J*L*Y*X*STAR*DENOM3
0243 BUB(9)=J*L*Y*Y*STAR*DENOM4
0244 DO 17 KK=1,9
0245 17 US4=US4+BUB(KK)
0246 BUB(1)=J*L*DENOM2
0247 BUB(2)=J*L*X*STAR*DENOM3
0248 BUB(3)=J*L*Y*STAR*DENOM4
0249 BUB(4)=J*L*X*DENOM3
0250 BUB(5)=J*L*X*X*STAR*DENOM4
0251 BUB(6)=J*L*X*Y*STAR*DENOM5
0252 BUB(7)=J*L*Y*DENOM4
0253 BUB(8)=J*L*Y*X*STAR*DENOM5
0254 BUB(9)=J*L*Y*Y*STAR*DENOM6
0255 DO 18 KK=1,9
0256 18 US5=US5+BUB(KK)
0257 BUB(1)=J*L*DENOM4
0258 BUB(2)=J*L*X*STAR*DENOM5
0259 BUB(3)=J*L*Y*STAR*DENOM6
0260 BUB(4)=J*L*X*DENOM5
0261 BUB(5)=J*L*X*X*STAR*DENOM6
0262 BUB(6)=J*L*X*Y*STAR*DENOM7
0263 BUB(7)=J*L*Y*DENOM6
0264 BUB(8)=J*L*Y*X*STAR*DENOM7
0265 BUB(9)=J*L*Y*Y*STAR*DENOM8
0266 DO 19 KK=1,9
0267 19 US6=US6+BUB(KK)
0268 BUB(1)=J*K*DENOM
0269 BUB(2)=J*(K+1)*X*DENOM1
0270 BUB(3)=J*(K+2)*Y*DENOM2
0271 BUB(4)=J*K*X*STAR*DENOM1
0272 BUB(5)=J*(K+1)*X*STAR*DENOM2
0273 BUB(6)=J*(K+2)*X*STAR*DENOM3
0274 BUB(7)=J*K*Y*STAR*DENOM2
0275 BUB(8)=J*(K+1)*Y*STAR*DENOM3
0276 BUB(9)=J*(K+2)*Y*STAR*DENOM4
0277 DO 20 KK=1,9
0278 20 US7A=US7A+BUB(KK)
0279 BUB(1)=L*I*DENOM
0280 BUB(2)=L*(I+1)*X*STAR*DENOM1
0281 BUB(3)=L*(I+2)*Y*STAR*DENOM2
0282 BUB(4)=L*I*X*DENOM1
0283 BUB(5)=L*(I+1)*X*STAR*DENOM2
0284 BUB(6)=L*(I+2)*X*Y*STAR*DENOM3
0285 BUB(7)=L*I*Y*DENOM2
0286 BUB(8)=L*(I+1)*Y*X*STAR*DENOM3
0287 BUB(9)=L*(I+2)*Y*STAR*DENOM4
0288 DO 21 KK=1,9
0289 21 US7B=US7B+BUB(KK)
0290 BUB(1)=J*K*DENOM2
0291 BUB(2)=J*(K+1)*X*DENOM2
0292 BUB(3)=J*(K+2)*Y*DENOM3
0293 BUB(4)=J*K*X*STAR*DENOM3
0294 BUB(5)=J*(K+1)*X*STAR*DENOM4
0295 BUB(6)=J*(K+2)*X*STAR*DENOM5
0296 BUB(7)=J*K*Y*STAR*DENOM4
0297 BUB(8)=J*(K+1)*Y*STAR*DENOM5
0298 BUB(9)=J*(K+2)*Y*STAR*DENOM6
0299 DO 22 KK=1,9
0300 22 US8A=US8A+BUB(KK)
0301 BUB(1)=L*I*DENOM2
0302 BUB(2)=L*(I+1)*X*STAR*DENOM3
0303 BUB(3)=L*(I+2)*Y*STAR*DENOM4
0304 BUB(4)=L*I*X*DENOM3
0305 BUB(5)=L*(I+1)*X*STAR*DENOM4
0306 BUB(6)=L*(I+2)*X*Y*STAR*DENOM5
0307 BUB(7)=L*I*Y*DENOM4
0308 BUB(8)=L*(I+1)*Y*X*STAR*DENOM5
0309 BUB(9)=L*(I+2)*Y*STAR*DENOM6
0310 DO 23 KK=1,9
0311 23 US8B=US8B+BUB(KK)
0312 BUB(1)=J*K*DENOM4
0313 BUB(2)=J*(K+1)*X*DENOM5
0314 BUB(3)=J*(K+2)*Y*DENOM6
0315 BUB(4)=J*K*X*STAR*DENOM5
0316 BUB(5)=J*(K+1)*X*STAR*DENOM6
0317 BUB(6)=J*(K+2)*X*STAR*DENOM7
0318 BUB(7)=J*K*Y*STAR*DENOM6
0319 BUB(8)=J*(K+1)*Y*STAR*DENOM7
0320 BUB(9)=J*(K+2)*Y*STAR*DENOM8
0321 DO 24 KK=1,9
0322 24 US9A=US9A+BUB(KK)
0323 BUB(1)=L*I*DENOM4
0324 BUB(2)=L*(I+1)*X*STAR*DENOM5
0325 BUB(3)=L*(I+2)*Y*STAR*DENOM6
0326 BUB(4)=L*I*X*DENOM5
0327 BUB(5)=L*(I+1)*X*STAR*DENOM6
0328 BUB(6)=L*(I+2)*X*Y*STAR*DENOM7
0329 BUB(7)=L*I*Y*DENOM6
0330 BUB(8)=L*(I+1)*Y*X*STAR*DENOM7
0331 BUB(9)=L*(I+2)*Y*STAR*DENOM8
0332 DO 25 KK=1,9
0333 25 US9B=US9B+BUB(KK)
0334 RETURN
0335 END

```


0001

0002

```

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(15,6),B(15,6),C(6,15),D(6,6),E(1,6),X(1,6)
1,F(6,6),G(6,6)
2,P(6),Q(6)

```

LEAST SQUARES EXTRAPOLATION PROGRAM

SEE APPENDIX C

THIS PROGRAM SOLVES THE MATRIX EQUATION
 $X(TRANS) = B(TRANS) * A * (A(TRANS) * A)^{-1}$
 THIS IS USED IN THE LEAST SQUARES EXTRAPOLATION TO
 DETERMINE THE MIN. VALUE OF K, THE BUCKLING COEF.
 SEE APPENDIX E.

LOAD THE MATRICIES A AND B TRANS.

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```

DO 10 I=1,15
READ(5,7) N,M,B(1,I)
5 FORMAT(2I3,F10.7)
AM=M
AN=N
A(1,I)=1.0
A(1,I)=100./AM/AM
A(1,I)=10000./AM/AM/AM/AM
A(1,I)=100./AN/AN
A(1,I)=10000./AN/AN/AN/AN
A(1,I)=10000./AN/AN/AM/AM

```

```

10 CONTINUE
WRITE(6,105)
WRITE(6,100) ((A(I,J),J=1,6),I=1,15)
WRITE(6,103) ((B(I,J),J=1,15),I=1,15)

```

CALCULATE THE TRANSPOSE OF MATRIX A AND STORE IN MATRIX C.

0018

0019

0020

```

CALL GMTRA(A,C,15,6)
WRITE(6,106)
WRITE(6,101) ((C(I,J),J=1,15),I=1,6)

```

CALCULATE THE PRODUCT A TRANS * A AND STORE IN MATRIX D.

0021

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```

CALL GMPRD(C,A,D,6,15,6)
WRITE(6,107)
WRITE(6,102) ((D(I,J),J=1,6),I=1,6)
DO 30 I=1,6
DO 30 J=1,6
30 F(I,J)=D(I,J)

```

CALCULATE THE INVERSE OF THE PRODUCT A TRANS * A. THAT IS D INVERSE AND STORE THE INVERSE IN D.

0027

0028

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```

CALL MINV(D,6,Z,P,Q)
WRITE(6,108)
WRITE(6,104) ((D(I,J),J=1,6),I=1,6)
WRITE(6,103) Z

```

CHECK A TRANS * A FOR SINGULARITY

0031

0032

```

CALL GMPRD(D,F,6,6,6)
WRITE(6,104) ((G(I,J),J=1,6),I=1,6)

```

CALCULATE THE PRODUCT B TRANS * A AND STORE IN E.

0033

0034

0035

```

CALL GMPRD(B,A,E,1,15,6)
WRITE(6,109)
WRITE(6,104) ((E(1,J),J=1,6)

```

CALCULATE THE PRODUCT (B TRANS * A) * (A TRANS * A) INVERSE AND STORE IN X.

0036

0037

0038

```

CALL GMPRD(E,D,X,1,6,6)
WRITE(6,110)
WRITE(6,104) ((X(1,J),J=1,6)

```

PRINT THE VALUES OF X.

0039

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```

DO 20 I=1,6
WRITE(6,7) I,X(1,I)
7 FORMAT(3H X(1I,3H )=,F15.9)
20 CONTINUE
100 FORMAT(6E15.5)
101 FORMAT( 5E15.5)
102 FORMAT(6E15.5)
103 FORMAT( 5E15.5)
104 FORMAT( 6E15.5)
105 FORMAT(1H0,' MATRIX A')
106 FORMAT(1H0,' MATRIX AT')
107 FORMAT(1H0,' MATRIX ATA')
108 FORMAT(1H0,' MATRIX (ATA)IN')
109 FORMAT(1H0,' MATRIX BTA')
110 FORMAT(1H0,' MATRIX RTA(ATAIN')
99 STOP
END

```

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13. ABSTRACT If an infinite flat elastic plate containing a circular hole is subjected to a loading which amounts to simple uniaxial tension at distances remote from the hole, the classical solution by Kirsch provides an evaluation of the stress components throughout the plate, provided the loading is sufficiently small. However if the tensile loading is progressively increased, there comes a point when Kirsch's solution becomes invalid, either through inelastic action at the most highly stressed regions in the plate, or through buckling of the plate from its original plane. The question of buckling under these circumstances has previously been discussed only by Danis, who dealt experimentally with finite plates, and by Pellett who performed a theoretical study of an infinite plate. The present thesis pinpoints and corrects some errors in Pellett's analysis, and leads to the result that buckling impends when the tensile stress reaches the value $S_{cr} = 1.720 E (t/a)^2$ where E denotes Young's modulus and t/a denotes the ratio of plate thickness to hole radius. This evaluation is for Poisson's ratio $\nu = 0.3$, a commonly used value, but evaluations are also made for other values of ν, indicating that the variation with respect to ν is quite small.			

14.

KEY WORDS

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LINK B

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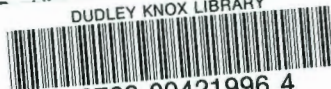
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